Solving Problems in ALGEBRA and TRIGONOMETRY

by

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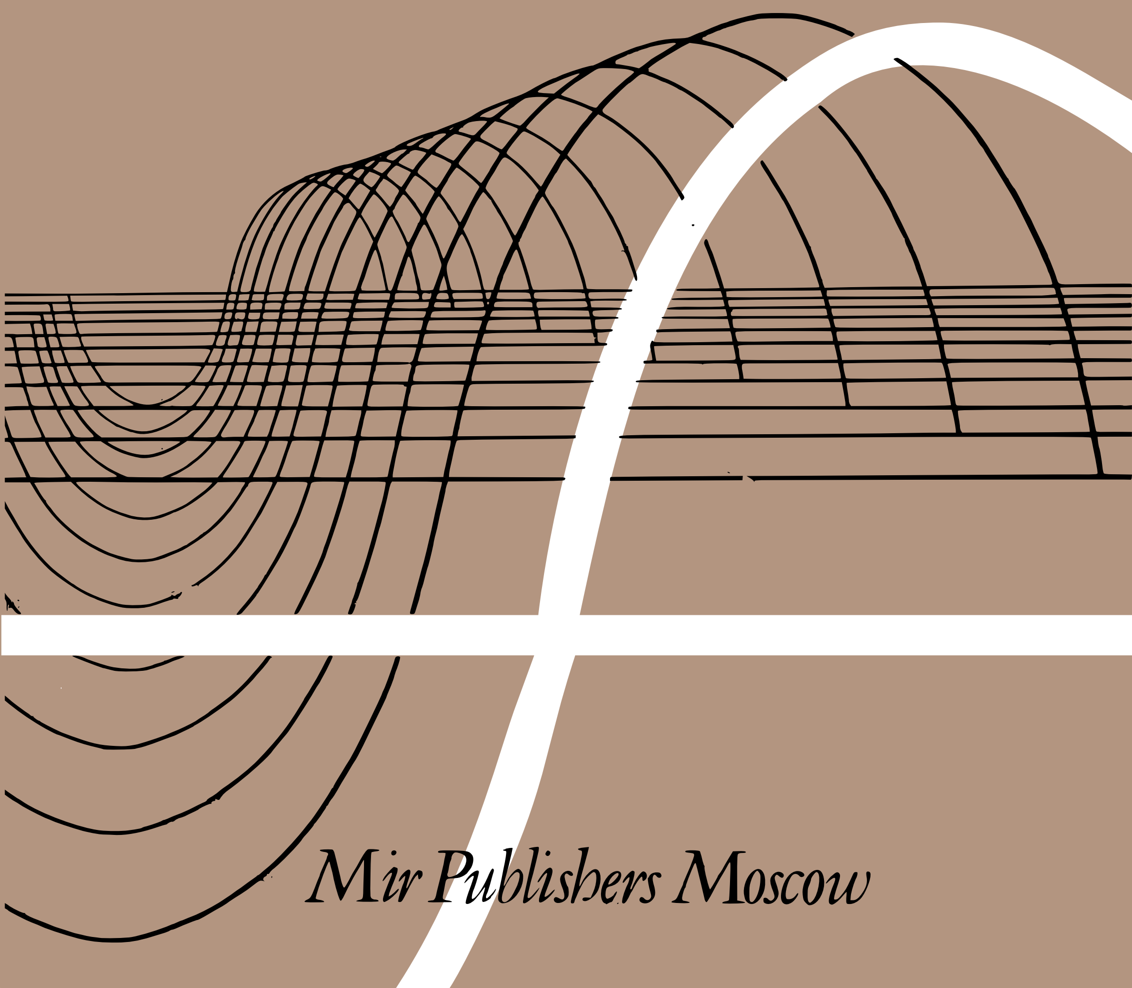

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В. Н. Литвиненко,
А. Г. Мордкович

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TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

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Preface

This study aid is intended for students of physical and mathematical faculties of pedagogical institutes.

The book contains about 2000 examples, problems, and exercises of which 1700 problems are for solving independently. Along with rather simple problems, there are also problems whose solution requires serious and sometimes inventive work. In the course of preparing the manuscript for print we tried to distribute the space among the basic types of "school" problems in algebra and trigonometry. Solving these problems will help the student to acquire professional skill necessary for a teacher who must know how to solve mathematical problems of the high-school level.

This book is not only a collection of problems, it is rather a study aid for practical work, as can be seen in the structure of the textbook. Each section contains necessary theoretical material and an ample number of worked examples (the total number of which amounts to about 300), which are very useful for the student primarily from the methodological point of view.

The present book is based on the series of our study aids designed for practical solving of mathematical problems published recently and intended for corresponding-course students. Various textbooks and study aids for schoolchildren, numerous books for teachers, various problem books in algebra, trigonometry, study aids for precollege students, problems for entrance examinations in mathematics, materials of school mathematical olympiads, etc., were also used in preparing the manuscript.

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The Authors

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Part I

ALGEBRA

Chapter 1

IDENTICAL TRANSFORMATIONS

SEC. 1. FACTORIZATION OF POLYNOMIALS

When solving many algebraic problems, it turns out to be necessary to represent a polynomial in the form of the product of two or more polynomials or as a polynomial and a monomial containing at least one variable. But not every polynomial is factorable over the field of real numbers. For example, it is impossible to factorize the polynomials $x + 3$ and $x^2 + 6x + 10$. Such polynomials are called *irreducible* or *prime*. The factorization of a polynomial is regarded to be completed if all the obtained factors are irreducible.

When factoring polynomials, we use various methods: taking out of the brackets a common factor, grouping, making use of the formulas for short-cut multiplication, and so on. Consider several examples to illustrate these methods.

Example 1. Factor the following polynomials:

(1) $f(a, b) = a^2 - 2a^3b - 2ab^3 + b^2$,

(2) $f(a) = a^3 - 7a^2 + 7a + 15$.

Solution. (1) Combining the extreme terms into one group and the middle terms into another, and taking out of the brackets a common factor in the second group, we get:

$$f(a, b) = (a^2 + b^2) - 2ab(a^2 + b^2) = (a^2 + b^2)(1 - 2ab).$$

(2) Let us represent the second and third terms of the given polynomial in the following way:

$$-7a^2 = -3a^2 - 4a^2; \quad 7a = 12a - 5a.$$

Then we write: $f(a) = a^3 - 3a^2 - 4a^2 + 12a - 5a + 15$. Grouping the terms pairwise and taking out of the brackets a common factor in each group, we get:

$$\begin{aligned} f(a) &= (a^3 - 3a^2) - (4a^2 - 12a) - (5a - 15) \\ &= a^2(a - 3) - 4a(a - 3) - 5(a - 3) \\ &= (a - 3)(a^2 - 4a - 5). \end{aligned}$$

It remains to factor the polynomial $a^2 - 4a - 5$. This can be done by the following two methods.

First Method. We have:

$$a^2 - 4a - 5 = a^2 + a - 5a - 5$$
$$= a(a+1) - 5(a+1) = (a+1)(a-5).$$

Second Method. From the equation $a^2 - 4a - 5 = 0$ we find the roots: $a_1 = -1$, $a_2 = 5$. Applying the formula for factoring the quadratic trinomial $ax^2 + bx + c = a(x - x_1)(x - x_2)$, we get:

$$a^2 - 4a - 5 = (a - a_1)(a - a_2) = (a+1)(a-5).$$

Thus,

$$f(a) = (a-3)(a+1)(a-5).$$

Example 2. Factor:

$$f(a, b, c) = ab(a+b) - bc(b+c) + ac(a-c).$$

Solution. We take advantage of the fact that the expression contained in the first parentheses is the sum of the expressions contained in the second and third parentheses: $a + b = (b + c) + (a - c)$. Then

$$f(a, b, c) = ab((b+c) + (a-c)) - bc(b+c) + ac(a-c)$$
$$= ab(b+c) + ab(a-c) - bc(b+c) + ac(a-c).$$

Grouping the terms and taking out of the brackets a common factor in each group, we get:

$$f(a, b, c) = (ab(b+c) - bc(b+c)) + (ab(a-c)$$
$$+ ac(a-c)) = (b+c)(ab-bc)$$
$$+ (a-c)(ab+ac) = (b+c)b(a-c)$$
$$+ (a-c)a(b+c) = (a-c)(b+c)(a+b).$$

Example 3. Factor:

$$f(a) = a^3 - 5a^2 - a + 5.$$

Solution. Grouping the terms and taking out of the brackets a common factor, we get:

$$f(a) = (a^3 - 5a^2) - (a-5) = a^2(a-5) - (a-5)$$
$$= (a-5)(a^2-1).$$

Using the formula $p^2 - q^2 = (p-q)(p+q)$, we get:

$$f(a) = (a-5)(a-1)(a+1).$$

Example 4. Factor:

$$f = (a, b) = 4a^2 - 12ab + 5b^2.$$

Solution. Completing the binomial $4a^2 - 12ab$ to a perfect square, we get: $(2a)^2 - 2(2a)(3b) + (3b)^2$. Then

$$\begin{aligned} f(a, b) &= (4a^2 - 12ab + 9b^2) - 9b^2 + 5b^2 \\ &= (2a - 3b)^2 - (2b)^2 = (2a - 3b - 2b)(2a - 3b + 2b) \\ &= (2a - 5b)(2a - b). \end{aligned}$$

Example 5. Factor:

$$f(a) = a^4 - 10a^2 + 169.$$

Solution. Noting that $a^4 + 169 = (a^2)^2 + 13^2$, and completing this sum to a perfect square, we get:

$$\begin{aligned} f(a) &= (a^4 + 26a^2 + 169) - 26a^2 - 10a^2 \\ &= (a^2 + 13)^2 - (6a)^2 = (a^2 - 6a + 13)(a^2 + 6a + 13). \end{aligned}$$

Example 6. Factor:

$$f(a, b) = a^6 + a^4 + a^2b^2 + b^4 - b^6.$$

Solution. Since

$$\begin{aligned} a^6 - b^6 &= (a^3)^2 - (b^3)^2 = (a^3 - b^3)(a^3 + b^3) \\ &= (a - b)(a^2 + ab + b^2)(a + b)(a^2 - ab + b^2) \end{aligned}$$

and

$$\begin{aligned} a^4 + a^2b^2 + b^4 &= (a^4 + 2a^2b^2 + b^4) - a^2b^2 = (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2), \end{aligned}$$

we have:

$$\begin{aligned} f(a, b) &= (a^2 + ab + b^2)(a^2 - ab + b^2)((a - b)(a + b) + 1) \\ &= (a^2 + ab + b^2)(a^2 - ab + b^2)(a^2 - b^2 + 1). \end{aligned}$$

Example 7. Factor:

$$f(a) = a^3 + 9a^2 + 27a + 19.$$

Solution. It is easy to see that, in order to obtain a perfect cube of the sum, the given function may be rewritten as follows:

$$\begin{aligned} f(a) &= (a^3 + 9a^2 + 27a + 27) - 8 = (a + 3)^3 - 2^3 \\ &= (a + 3 - 2)((a + 3)^2 + (a + 3) \times 2 + 4) \\ &= (a + 1)(a^2 + 8a + 19). \end{aligned}$$

Example 8. Prove that if $a \in N$ and $f(a) = a^4 + 6a^3 + 11a^2 + 6a$, then $f(a) \vdots 24$*.

* The symbol $\vdots$ means "is divisible by" (without a remainder).

Solution. Represent $6a^3$ and $11a^2$ as sums of like terms: $6a^3 = a^3 + 5a^3$ and $11a^2 = 5a^2 + 6a^2$. Then

$$\begin{aligned} f(a) &= a^4 + (a^3 + 5a^3) + (5a^2 + 6a^2) + 6a \\ &= (a^4 + a^3) + (5a^3 + 5a^2) + (6a^2 + 6a) \\ &= a^3(a+1) + 5a^2(a+1) + 6a(a+1) \\ &= a(a+1)(a^2 + 5a + 6) = a(a+1)(a+2)(a+3). \end{aligned}$$

But of four successive natural numbers at least one is divisible by 3, and two numbers are even, that is, one of them is divisible by 4, and, hence, the product of these four numbers is divisible by the product $3 \times 2 \times 4$. Thus, $f(a) \vdots 24$.

Example 9. Prove that if $f(a) = a^2(a^2 + 14) + 49$, where a is an odd number, then $f(a) \vdots 64$.

Solution. Note that $f(a) = a^4 + 14a^2 + 49 = (a^2 + 7)^2$. Since a is odd, we have: $a = 2n - 1$, where $n \in N$. Then $f(a) = f(2n - 1) = ((2n - 1)^2 + 7)^2 = (4n^2 - 4n + 8)^2 = 16(n^2 - n + 2)^2$. The obtained expression is divisible by 16. Therefore, to prove that $f(a) \vdots 64$, it is sufficient to show that $(n^2 - n + 2)^2 \vdots 4$. Consider two possible cases: (1) n is an even number and (2) n is an odd number.

(1) If n is even, then n^2 is also even and, consequently, $n^2 - n + 2$ is even, that is, $(n^2 - n + 2) \vdots 2$, therefore $(n^2 - n + 2)^2 \vdots 4$, and, hence, $f(a) \vdots 64$.

(2) If n is odd, then n^2 is also odd, but then $n^2 - n$ is even and $n^2 - n + 2$ is also even. Thus, in this case also $f(a) \vdots 64$.

EXERCISES

In Problems 1 through 44, factor the given expressions:

1. $a^4 - 1$. **2.** $a^6 - 1$. **3.** $a^6 + 1$. **4.** $a^4 - 18a^2 + 81$.
5. $a^{12} - 2a^6 + 1$. **6.** $a^5 + a^3 - a^2 - 1$. **7.** $a^4 + 2a^3 - 2a - 1$.
8. $4b^2c^2 - (b^2 + c^2 - a^2)^2$.
9. $a^4 + a^2b^2 + b^4$. **10.** $a^4 + 4a^2 - 5$.
11. $4a^4 + 5a^2 + 1$. **12.** $c^4 - (1 + ab)c^2 + ab$.
13. $a^4 + 324$. **14.** $a^4 + a^2 + 1$.
15. $a^8 + a^4 + 1$. **16.** $2a^4 + a^3 + 4a^2 + a + 2$.
17. $a^4 + 3a^3 + 4a^2 - 6a - 12$.
18. $(a^2 + a + 3)(a^2 + a + 4) - 12$. **19.** $a^5 + a^3 - a^2 - 1$.
20. $2a^2b + 4ab^2 - a^2c + ac^2 - 4b^2c + 2bc^2 - 4abc$.
21. $(ab + ac + bc)(a + b + c) - abc$.
22. $a(b - 2c)^2 + b(a - 2c)^2 - 2c(a + b)^2 + 8abc$.
23. $a^3(a^2 - 7)^2 - 36a$. **24.** $(a + b)^5 - (a^5 + b^5)$.
25. $a^2b^2(b - a) + b^2c^2(c - b) + a^2c^2(a - c)$.
26. $8a^3(b + c) - b^3(2a + c) - c^3(2a - b)$.
27. $(a + b + c)^3 - (a^3 + b^3 + c^3)$.
28. $a^4 + 9$. **29.** $a^4 + b^4$.
30. $a^3 + 5a^2 + 3a - 9$. **31.** $a(a + 1)(a + 2)(a + 3) + 1$.
32. $(a + 1)(a + 3)(a + 5)(a + 7) + 15$.
33. $2(a^2 + 2a - 1)^2 + 5(a^2 + 2a - 1)(a^2 + 1) + 2(a^2 + 1)^2$.
34. $(a - b)c^3 - (a - c)b^3 + (b - c)a^3$.
35. $(a - b)^3 + (b - c)^3 - (a - c)^3$.

36. $(a^2 + b^2)^3 - (b^2 + c^2)^3 - (a^2 - c^2)^3$.
37. $a^4 + 2a^3b - 3a^2b^2 - 4ab^3 - b^4$.
38. $a^2b + ab^2 + a^2c + b^2c + bc^2 + 3abc$.
39. $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2$.
40. $a^5 + a^4 + a^3 + a^2 + a + 1$. 41. $a^4 + 2a^3 + 3a^2 + 2a + 1$.
42. $a^4 - 2a^3b - 8a^2b^2 - 6ab^3 - b^4$.
43. $a^4 + a^2 + \sqrt{2a} + 2$. 44. $a^{10} + a^5 + 1$.
45. Prove that if $a \in N$, then $(a^5 - 5a^3 + 4a) \vdots 120$.
46. Prove that if a is a number relatively prime with respect to 6, then $(a^2 - 1) \vdots 24$.
47. Prove that if $a \in N$, then $(2a^3 + 3a^2 + a) \vdots 6$.
48. For what values of $a \in N$ is the expression $a^4 + 4$ a prime number?
49. Prove that if a is even, then $\frac{a}{12} + \frac{a^2}{8} + \frac{a^3}{24}$ is a whole number.
50. Prove that if $a \in N$, then $\frac{a^5}{120} + \frac{a^4}{12} + \frac{7a^3}{24} + \frac{5a^2}{12} + \frac{a}{5}$ is a whole number.

SEC. 2. IDENTICAL TRANSFORMATIONS OF RATIONAL FUNCTIONS

The replacement of an analytic function with another which is identical to it on a certain set is called an *identical transformation* of the given function on this set.

Identical transformations of a function may change its domain of definition. Thus, when collecting like terms in the course of simplifying the function

$$x^2 + 3x - 5 + \sqrt{x} - \sqrt{x}, \tag{1}$$

we extend its domain of definition: the given function is defined only for $x \geqslant 0$, whereas the polynomial

$$x^2 + 3x - 5 \tag{2}$$

obtained as the result of the simplification is defined for any value of x. Functions (1) and (2) are identical only on the set $[0, \infty)$.

The domain of definition of a function may also change after reducing a fraction. Thus, the algebraic fraction

$$\frac{x^3 - 1}{(x-1)(x+2)} \tag{3}$$

is defined for $x \neq 1$, $x \neq -2$. On reducing by $x - 1$ we get the fraction

$$\frac{x^2 + x + 1}{x + 2}, \tag{4}$$

which is defined for $x \neq -2$. Functions (3) and (4) are identical on the set $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

A change in the domain of definition of a function may also occur as a result of some other transformations; therefore, after a given function is transformed, one should be able to indicate the set where the given function is identical to the obtained one.

An algebraic function is called *rational* if it contains only the operations of addition, multiplication, subtraction, division, and raising to an integer power.

Example 1. Simplify the function $f(a, b) = \dfrac{2a^2 + ab - b^2}{a + b}$.

Solution. Representing ab as the sum of like terms $2ab - ab$, we get:

$$2a^2 + ab - b^2 = 2a^2 + 2ab - ab - b^2 = 2a(a + b) - b(a + b)$$
$$= (a + b)(2a - b).$$

Then

$$f(a, b) = \frac{(a+b)(2a-b)}{a+b} = 2a - b.$$

Since the reduction by $a + b$ can be performed only if $a + b \neq 0$, $f(a, b) = 2a - b$ if $a \neq -b$.

Example 2. Simplify the function $f(a) = \dfrac{a^4 - 10a^2 + 169}{a^2 + 6a + 13}$.

Solution. Factoring the numerator, we get (see Example 5 in the preceding section): $a^4 - 10a^2 + 169 = (a^2 + 6a + 13) \times (a^2 - 6a + 13)$.

Hence,

$$f(a) = \frac{(a^2+6a+13)(a^2-6a+13)}{a^2+6a+13} = a^2 - 6a + 13.$$

Since $a^2 + 6a + 13$ does not vanish for any real value of a (indeed, $a^2 + 6a + 13 = (a + 3)^2 + 4 > 0$), we have: $f(a) = a^2 - 6a + 13$ for all values of a.

Example 3. Simplify the function

$$f(a) = \left(\frac{1}{a^2+3a+2} + \frac{2a}{a^2+4a+3} + \frac{1}{a^2+5a+6}\right)^2 \frac{(a-3)^2+12a}{2}.$$

Solution. Performing the above operations, we get:

$$f(a) = \left(\frac{a+3+2a(a+2)+a+1}{(a+1)(a+2)(a+3)}\right)^2 \frac{a^2-6a+9+12a}{2}$$
$$= \left(\frac{2a^2+6a+4}{(a+1)(a+2)(a+3)}\right)^2 \frac{a^2+6a+9}{2}$$
$$= 4\left(\frac{a^2+3a+2}{(a^2+3a+2)(a+3)}\right)^2 \frac{(a+3)^2}{2} = 2.$$

Thus, $f(a) = 2$ if $a \neq -1$, $a \neq -2$, $a \neq -3$.

Example 4. Simplify the function

$$f(a, b, c) = \frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-c)(b-a)} + \frac{c^2}{(c-a)(c-b)}.$$

Solution. Reducing all the fractions to a least common denominator, we get:

$$f(a, b, c) = \frac{a^2(b-c) - b^2(a-c) + c^2(a-b)}{(a-b)(b-c)(a-c)}.$$

Noticing that $b - c = (a - c) - (a - b)$, we transform the numerator in the following way:

$$\begin{aligned} a^2(b-c) &- b^2(a-c) + c^2(a-b) \\ &= a^2(a-c) - a^2(a-b) - b^2(a-c) + c^2(a-b) \\ &= (a-c)(a^2-b^2) + (a-b)(c^2-a^2) \\ &= (a-c)(a-b)(a+b-c-a) \\ &= (a-b)(b-c)(a-c). \end{aligned}$$

Thus, $f(a, b, c) = 1$ if $a \neq b$, $b \neq c$, $a \neq c$.

Example 5. Prove that if $a + b + c = 0$, then

$$a^3 + b^3 + c^3 = 3abc.$$

Solution. Since $a + b + c = 0$, then $a = -b - c$. Then

$$\begin{aligned} a^3 + b^3 + c^3 &= (-b-c)^3 + b^3 + c^3 = -(b+c)^3 + b^3 + c^3 \\ &= -(b^3 + 3b^2c + 3bc^2 + c^3) + b^3 + c^3 = -(3b^2c + 3bc^2) \\ &= -3bc(b+c). \end{aligned}$$

But $b + c = -a$. Thus, $a^3 + b^3 + c^3 = -3bc(-a) = 3abc$.

Example 6. Prove that if $a + b + c = 0$, where $a \neq 0$, $b \neq 0$, $c \neq 0$, then

$$\left(\frac{a-b}{c} + \frac{b-c}{a} + \frac{c-a}{b}\right)\left(\frac{c}{a-b} + \frac{a}{b-c} + \frac{b}{c-a}\right) = 9.$$

Solution. Consider the product of the first multiplier and the first fraction of the second multiplier:

$$\begin{aligned} \left(\frac{a-b}{c} + \frac{b-c}{a} + \frac{c-a}{b}\right)\frac{c}{a-b} &= 1 + \left(\frac{b-c}{a} + \frac{c-a}{b}\right)\frac{c}{a-b} \\ = 1 + \frac{b^2 - bc + ac - a^2}{ab}\,\frac{c}{a-b} &= 1 + \frac{c(a-b) - (a^2-b^2)}{ab}\,\frac{c}{a-b} \\ = 1 + \frac{(a-b)(c-(a+b))}{ab}\,\frac{c}{a-b} &= 1 + \frac{c}{ab}(c-(a+b)). \end{aligned}$$

But, by the hypothesis, $a + b = -c$. Therefore for the product under consideration we get: $1 + \frac{2c^2}{ab}$.

Similarly, the product of the first multiplier by the second fraction of the second multiplier is equal to $1+\frac{2a^2}{bc}$, and the product by the third fraction is equal to $1+\frac{2b^2}{ca}$. Adding together the obtained results, we get:

$$1+\frac{2c^2}{ab}+1+\frac{2a^2}{bc}+1+\frac{2b^2}{ca}=3+2\left(\frac{c^2}{ab}+\frac{a^2}{bc}+\frac{b^2}{ac}\right)$$
$$=3+\frac{2(c^3+a^3+b^3)}{abc}.$$

Since $a^3+b^3+c^3=3abc$ (see Example 5), we have:

$$3+\frac{2(a^3+b^3+c^3)}{abc}=3+\frac{2\times 3abc}{abc}=9,$$

which was required to be proved.

In the following examples the identical transformations of rational functions serve as a means of solving problems using the method of mathematical induction.

The method of mathematical induction is formulated as follows: *A statement depending on a natural number n holds true for any n if the following two conditions are fulfilled*:

(a) *the statement is true for* $n = 1$;

(b) *the validity of the statement for* $n = k$ (*for any natural value of* k) *implies its validity also for* $n = k + 1$.

The proof by the method of mathematical induction is carried out in the following way. First, the statement being proved is verified for $n = 1$. This part of the proof is called the *basis* of induction. The next part of the proof is termed the *induction step*. It proves the validity of the statement for $n = k + 1$ in the assumption of the validity of the statement for $n = k$ (the assumption of induction).

Example 7. Prove that

$$1^2+2^2+3^3+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}$$

Solution. For $n = 1$ the statement is true since

$$1^2=\frac{1(1+1)(2+1)}{6}.$$

Suppose that it is true for $n = k$, that is,

$$1^2+2^2+3^2+\ldots+k^2=\frac{k(k+1)(2k+1)}{6}.$$

Let us prove that it is also true for $n = k + 1$, that is,

$$1^2+2^2+3^2+\ldots+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}.$$

Indeed,

$$\begin{aligned}&1^2+2^2+3^2+\ldots+k^2+(k+1)^2\\&=\frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}\\&=\frac{(k+1)(2k^2+7k+6)}{6}=\frac{(k+1)(k+2)(2k+3)}{6}.\end{aligned}$$

Thereby we have proved that the statement is true for any natural number n.

Example 8. Prove that $1^3+2^3+3^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$.

Solution. For $n=1$ the statement is true since $1^3=\left(\frac{1(1+1)}{2}\right)^2$. Suppose that it is true for $n=k$, that is, $1^3+2^3+3^3+\ldots+k^3=\left(\frac{k(k+1)}{2}\right)^2$. Let us prove that then it is also true for $n=k+1$, that is,

$$1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\left(\frac{(k+1)(k+2)}{2}\right)^2.$$

Indeed,

$$\begin{aligned}&1^3+2^3+3^3+\ldots+k^3+(k+1)^3=\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3\\&=\frac{(k(k+1))^2+4(k+1)^3}{4}=\frac{(k+1)^2(k^2+4k+4)}{4}=\left(\frac{(k+1)(k+2)}{2}\right)^2.\end{aligned}$$

Thereby we have proved that the statement is true for any natural number n.

Example 9. Prove that the sum of the cubes of three successive natural numbers is divisible by 9.

Solution. Let us prove that

$$(n^3+(n+1)^3+(n+2)^3) \vdots 9 \tag{5}$$

for any natural n. Let us, first of all, verify whether the statement (5) is true for $n=1$. We have: $1^3+2^3+3^3=36$, but $36 \vdots 9$, consequently, for $n=1$ the statement is true.

Suppose that the statement (5) is true for $n=k$, that is,

$$(k^3+(k+1)^3+(k+2)^3) \vdots 9.$$

Let us prove that it is also true for $n=k+1$. Indeed, $(k+1)^3+(k+2)^3+(k+3)^3=(k+1)^3+(k+2)^3+k^3+9k^2+27k+27=(k^3+(k+1)^3+(k+2)^3)+9(k^2+3k+3)$. Since each term of the obtained sum is divisible by 9 (the first term by virtue of the assumption of induction, the second one as containing the multiplier 9), the sum is also divisible by 9. Applying the principle of mathematical induction, we conclude that the statement is true for all $n \in N$.

Example 10. Prove that

$$(3^{2n+1} + 40n - 67) \vdots 64 \tag{6}$$

for any natural n.

Solution. If $n = 1$, then $3^3 + 40 \times 1 - 67 = 0$. But $0 \vdots 64$, hence, for $n = 1$ the statement (6) is true. Let us suppose that it is true for $n = k$, that is, $(3^{2k+1} + 40k - 67) \vdots 64$. Let us prove that then it is also true for $n = k + 1$. Indeed, we have: $3^{2k+3} + 40(k + 1) - 67 = 9 \times 3^{2k+1} + 40k - 27 = 9(3^{2k+1} + 40k - 67) - 320k + 576 = 9(3^{2k+1} + 40k - 67) + 64(9 - 5k)$.

Each of the terms is divisible by 64, consequently, the entire sum is also divisible by 64. Thus, the statement (6) is true for all $n \in N$.

Example 11. Prove that

$$(n^4 + 6n^3 + 11n^2 + 6n) \vdots 24 \tag{7}$$

for any natural n.

Solution. For $n = 1$ the statement is true since $1 + 6 + 11 + 6 = 24$, and $24 \vdots 24$.

Suppose that the statement (7) is true for $n = k$, that is, $(k^4 + 6k^3 + 11k^2 + 6k) \vdots 24$. Let us prove that then it is also true for $n = k + 1$. Indeed, we have: $(k + 1)^4 + 6(k + 1)^3 + 11(k + 1)^2 + 6(k+1) = (k^4 + 6k^3 + 11k^2 + 6k) + 24(k^2 + 1) + 4(k^3 + 11k)$.

If we now prove that

$$(k^3 + 11k) \vdots 6 \tag{8}$$

for all k, thereby it will be proved that the given expression is divisible by 24. And here we are posed by a new problem which we are going to solve using the method of mathematical induction once again.

Let us first of all check whether the statement (8) is true for $k = 1$. This is obvious: $(1 + 11) \vdots 6$. Let the statement (8) be true for $k = m$, that is, $(m^3 + 11m) \vdots 6$. Let us prove that it is then true for $k = m + 1$. Indeed,

$$(m + 1)^3 + 11(m + 1) = (m^3 + 11m) + 12 + 3m(m + 1).$$

Of the two successive natural numbers m and $(m + 1)$, one is necessarily even, hence $(m(m + 1)) \vdots 2$, and $(3m(m + 1)) \vdots 6$. But then $((m^3 + 11m) + 12 + 3m(m + 1)) \vdots 6$.

Hence, we conclude that $(k^3 + 11k) \vdots 6$ for any natural k. The statement (8) has been proved. Thus, the statement (7) is true for all $n \in N$.

Note that the considered example can be solved without applying the method of mathematical induction.

EXERCISES

In Problems 51 through 57, reduce the given fractions:

51. $\dfrac{5a^2-a-4}{a^3-1}$. 52. $\dfrac{a^6+a^4+a^2+1}{a^3+a^2+a+1}$.

53. $\dfrac{a^4+a^2-2}{a^6+8}$. 54. $\dfrac{a^4-a^2-12}{a^4+8a^2+15}$.

55. $\dfrac{2a^4+7a^2+6}{3a^4+3a^2-6}$. 56. $\dfrac{5a^4+5a^2-3a^2b-3b}{a^4+3a^2+2}$. 57. $\dfrac{a^4+a^2b^2+b^4}{a^6-b^6}$.

In Problems 58 through 70, simplify the indicated functions:

58. $\dfrac{1}{1-a}-\dfrac{1}{1+a}-\dfrac{2a}{1+a^2}-\dfrac{4a^3}{1+a^4}-\dfrac{8a^7}{1+a^8}$.

59. $\dfrac{1}{1-a}+\dfrac{1}{1+a}+\dfrac{2}{1+a^2}+\dfrac{4}{1+a^4}+\dfrac{8}{1+a^8}+\dfrac{16}{1+a^{16}}$.

60. $\dfrac{1}{a(a+1)}+\dfrac{1}{(a+1)(a+2)}+\dfrac{1}{(a+2)(a+3)}+\dfrac{1}{(a+3)(a+4)}$
$+\dfrac{1}{(a+4)(a+5)}$.

61. $\dfrac{a}{a^2-1}+\dfrac{a^2+a-1}{a^3-a^2+a-1}+\dfrac{a^2-a-1}{a^3+a^2+a+1}-\dfrac{2a^3}{a^4-1}$.

62. $\left(\dfrac{b}{a+b}+a\right)\left(\dfrac{a}{a-b}-b\right)-\left(\dfrac{a}{a+b}+b\right)\left(\dfrac{b}{a-b}-a\right)$.

63. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\left(1+\dfrac{b^2+c^2-a^2}{2bc}\right)$.

64. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}$.

65. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}$.

66. $\dfrac{a-c}{a^2+ac+c^2}\,\dfrac{a^3-c^3}{a^2b-bc^2}\left(1+\dfrac{c}{a-c}-\dfrac{1+c}{c}\right)\div\dfrac{c(1+c)-a}{bc}$.

67. $\dfrac{\dfrac{a}{8b^3}+\dfrac{1}{4b^2}}{a^2+2ab+2b^2}-\dfrac{\dfrac{a}{8b^3}-\dfrac{1}{4b^2}}{a^2-2ab+2b^2}-\dfrac{1}{4b^2(a^2+2b^2)}+\dfrac{1}{4b^2(a^2-2b^2)}$.

68. $\dfrac{a-b}{a+b}+\dfrac{b-c}{b+c}+\dfrac{c-a}{c+a}+\dfrac{(a-b)(b-c)(c-a)}{(a+b)(b+c)(c+a)}$.

69. $\dfrac{a^3b-ab^3+b^3c-bc^3+c^3a-ca^3}{a^2b-ab^2+b^2c-bc^2+c^2a-ca^2}$.

70. $\dfrac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}$.

In Problems 71 and 72, prove the given identities:

71. $\dfrac{b-c}{(a-b)(a-c)}+\dfrac{c-a}{(b-c)(b-a)}+\dfrac{a-b}{(c-a)(c-b)}=\dfrac{2}{a-b}+\dfrac{2}{b-c}+\dfrac{2}{c-a}$

72. $a^2\frac{(d-b)(d-c)}{(a-b)(a-c)}+b^2\frac{(d-c)(d-a)}{(b-c)(b-a)}+c^2\frac{(d-a)(d-b)}{(c-a)(c-b)}=d^2.$

73. Prove that if $a, b, c \in \boldsymbol{R}$, then the equality $(a-b)^2+(b-c)^2+(c-a)^2=(a+b-2c)^2+(b+c-2a)^2+(c+a-2b)^2$ implies: $a=b=c$.

74. Prove that $(a-1)(a-3)(a-4)(a-6)+10$ is a positive number for $a \in \boldsymbol{R}$.

75. Find the least value of the function $(a-1)(a-3)(a-4)(a-6)+10$.

76. Prove that if $a+b+c=0$, then
$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2}.$$

77. Prove that if $a+b+c=0$, then
$$\frac{a^7+b^7+c^7}{7}=\frac{a^5+b^5+c^5}{5}\cdot\frac{a^2+b^2+c^2}{2}.$$

78. Prove that if $\frac{l}{a}+\frac{m}{b}+\frac{n}{c}=1$ and $\frac{a}{l}+\frac{b}{m}+\frac{c}{n}=0$, then $\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}=1$.

79. Prove that if $\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}=0$, where $a \neq b$, $a \neq c$, $b \neq c$, then $\frac{a}{(b-c)^2}+\frac{b}{(c-a)^2}+\frac{c}{(a-b)^2}=0$.

80. Prove that if $a+b+c=0$, then $a^5(b^2+c^2)+b^5(a^2+c^2)+c^5(b^2+a^2)=\frac{(a^3+b^3+c^3)(a^4+b^4+c^4)}{2}$.

In Problems 81 through 96, prove the given identities using the method of mathematical induction. *

81. $1\times 2+2\times 3+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}.$

82. $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\ldots\left(1-\frac{1}{n+1}\right)=\frac{1}{n+1}.$

83. $1\times 4+2\times 7+3\times 10+\ldots+n(3n+1)=n(n+1)^2.$

84. $\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\ldots\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+2}{2n+2}.$

85. $1\times 1!+2\times 2!+\ldots+n\times n!=(n+1)!-1.$

86. $\frac{0}{1!}+\frac{1}{2!}+\frac{2}{3!}+\ldots+\frac{n-1}{n!}=1-\frac{1}{n!}.$

87. $\frac{1^2}{1\times 3}+\frac{2^2}{3\times 5}+\ldots+\frac{n^2}{(2n-1)(2n+1)}=\frac{n(n+1)}{2(2n+1)}.$

88. $\frac{1}{1\times 3\times 5}+\frac{2}{3\times 5\times 7}+\ldots+\frac{n}{(2n-1)(2n+1)(2n+3)}=\frac{n(n+1)}{2(2n+1)(2n+3)}.$

89. $\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right).$

* In Problems 81 through 119, it is assumed that $n \in \boldsymbol{N}$.

90. $1\times 2\times 3+2\times 3\times 4+\ldots+n(n+1)(n+2)=\dfrac{n(n+1)(n+2)(n+3)}{4}$.

91. $2\times 1^2+3\times 2^2+\ldots+(n+1)\,n^2=\dfrac{n(n+1)(n+2)(3n+1)}{12}$.

92. $\dfrac{1}{1\times 2\times 3\times 4}+\dfrac{1}{2\times 3\times 4\times 5}+\ldots+\dfrac{1}{n(n+1)(n+2)(n+3)}$
$=\dfrac{1}{3}\left(\dfrac{1}{6}-\dfrac{1}{(n+1)(n+2)(n+3)}\right)$.

93. $1+x+x^2+\ldots+x^n=\dfrac{x^{n+1}-1}{x-1}$, where $x\neq 1$.

94. $7+77+777+\ldots+\underbrace{777\ldots7}_{n\text{ digits}}=\dfrac{7(10^{n+1}-9n-10)}{81}$.

95. $(n+1)(n+2)\ \ldots\ (n+n)=2^n\times 1\times 3\times 5\times\ldots\times(2n-1)$.

96. $1-\dfrac{1}{2}+\dfrac{1}{3}-\dfrac{1}{4}+\ldots+\dfrac{1}{2n-1}-\dfrac{1}{2n}=\dfrac{1}{n+1}+\ldots+\dfrac{1}{2n}$.

In Problems 97 through 101, derive formulas for the given sums:

97. $S_n=\dfrac{1}{1\times 3}+\dfrac{1}{3\times 5}+\ldots+\dfrac{1}{(2n-1)(2n+1)}$.

98. $S_n=\dfrac{1}{1\times 4}+\dfrac{1}{4\times 7}+\ldots+\dfrac{1}{(3n-2)(3n+1)}$.

99. $S_n=\dfrac{1}{1\times 5}+\dfrac{1}{5\times 9}+\ldots+\dfrac{1}{(4n-3)(4n+1)}$.

100. $S_n=\dfrac{1}{1\times 6}+\dfrac{1}{6\times 11}+\ldots+\dfrac{1}{(5n-4)(5n+1)}$.

101. $S_n=1^2-2^2+3^2-4^2+\ldots+(-1)^{n-1}n^2$.

In Problems 102 through 106, prove the given identities:

102. $x+2x^2+3x^3+\ldots+nx^n=\dfrac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$, where $x\neq 1$.

103. $\dfrac{a+1}{2}+\dfrac{a+3}{4}+\dfrac{a+7}{8}+\ldots+\dfrac{a+2^n-1}{2^n}=\dfrac{(a-1)(2^n-1)}{2^n}$.

104. $\dfrac{1}{1+x}+\dfrac{2}{1+x^2}+\dfrac{4}{1+x^4}+\ldots+\dfrac{2^n}{1+x^{2^n}}=\dfrac{1}{x-1}+\dfrac{2^{n+1}}{1-x^{2^{n+1}}}$, where $|x|\neq 1$.

105. $\dfrac{x}{1-x^2}+\dfrac{x^2}{1-x^4}+\dfrac{x^4}{1-x^8}+\ldots+\dfrac{x^{2^{n-1}}}{1-x^{2^n}}=\dfrac{1}{1-x}\,\dfrac{x-x^{2^n}}{1-x^{2^n}}$, where $|x|\neq 1$.

106. $\left(x-\dfrac{1}{x}\right)^2+\left(x^2-\dfrac{1}{x^2}\right)^2+\ldots+\left(x^n-\dfrac{1}{x^n}\right)^2=\dfrac{1}{x^2-1}\left(x^{2n+2}-\dfrac{1}{x^{2n}}\right)$
$-2n-1$.

In Problems 107 through 119, prove that the given statements are true:

107. $(6^{2n}-1) \vdots 35$. 108. $(4^n+15n-1) \vdots 9$.
109. $(2^{5n+3}+5^n \times 3^{n+2}) \vdots 17$. 110. $(6^{2n}+3^{n+2}+3^n) \vdots 11$.
111. $(3^{2n+2}-8n-9) \vdots 64$. 112. $(3^{3n+2}+5 \times 2^{3n+1}) \vdots 19$.
113. $(2^{n+5} \times 3^{4n}+5^{3n+1}) \vdots 37$.
114. $(7^{n+2}+8^{2n+1}) \vdots 57$. 115. $(11^{n+2}+12^{2n+1}) \vdots 133$.
116. $(2^{n+2} \times 3^n+5n-4) \vdots 25$. 117. $(5^{2n+1}+2^{n+4}+2^{n+1}) \vdots 23$.
118. $(3^{2n+2} \times 5^{2n}-3^{3n+2} \times 2^{2n}) \vdots 1053$.
119. $(n^6-3n^5+6n^4-7n^3-2n) \vdots 24$.

SEC. 3. IDENTICAL TRANSFORMATIONS OF IRRATIONAL FUNCTIONS

An algebraic function involving the extraction of the root of the variable or raising the latter to a noninteger rational power is said to be *irrational* with respect to this variable.

Let us recall the definition of the arithmetic root. If $a \geqslant 0$ and $n \in N$, $n > 1$, then there is only one nonnegative number x such that the equality $x^n = a$ is fulfilled. This number x is called the *n-th arithmetic root* of the nonnegative number a and is symbolized by $\sqrt[n]{a}$.

From the foregoing it follows that the equality $\sqrt{49} = 7$ is true, while the equalities $\sqrt{49} = -7$ or $\sqrt{49} = \pm 7$ are not true.

If n is an odd number exceeding 1, and $a < 0$, then $\sqrt[n]{a}$ is understood to be a negative number x such that $x^n = a$.

If n, k, $m \in N$, $a \geqslant 0$ and $b \geqslant 0$, then:

1°. $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.

This property is extended to the product of any number of factors, for instance, $\sqrt[3]{8 \times 27 \times 125} = \sqrt[3]{8} \times \sqrt[3]{27} \times \sqrt[3]{125} = 2 \times 3 \times 5 = 30$.

2°. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$ if $b \neq 0$.

Remark. If $a < 0$ and $b < 0$, then Properties 1° and 2° take the form:

$$\sqrt[n]{ab} = \sqrt[n]{|a|}\,\sqrt[n]{|b|};\quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{|a|}}{\sqrt[n]{|b|}}.$$

3°. $(\sqrt[n]{a})^k = \sqrt[n]{a^k}$, for instance, $(\sqrt[5]{a^2})^3 = \sqrt[5]{(a^2)^3} = \sqrt[5]{a^6}$.

4°. $\sqrt[n]{\sqrt[k]{a}} = \sqrt[nk]{a}$, for instance, $\sqrt[4]{\sqrt[3]{a}} = \sqrt[12]{a}$.

5°. $\sqrt[mn]{a^{mk}} = \sqrt[n]{a^k}$, for instance, $\sqrt[6]{a^4} = \sqrt[3]{a^2}$, $\sqrt[5]{a} = \sqrt[15]{a^3}$.

Remark. If the indices of roots are odd numbers, then Properties 1°-5° are fulfilled both for $a < 0$, $b < 0$, and for $ab < 0$.

Let us recall another important property of the arithmetic root: if n is an even number, that is, $n = 2k$, then the following identity takes place: $\sqrt[2k]{a^{2k}} = |a|$, for instance, $\sqrt{(\sqrt{3}-2)^2} = |\sqrt{3} - 2| = 2 - \sqrt{3}$.

Let us also recall the definition of a power with a rational exponent.

(1) If $a \neq 0$, then $a^0 = 1$.

(2) If $a \geqslant 0$, then $a^{m/n} = \sqrt[n]{a^m}$ (n, m natural numbers, $n \geqslant 2$).

(3) If $a > 0$, then $a^{-r} = \frac{1}{a^r}$ (r a positive rational number).

(4) If $a < 0$, $m \in \mathbf{Z}$, then $a^{-m} = a^{1/m}$.

The basic properties of powers with arbitrary rational exponents are listed below:

1°. $a^r \times a^s = a^{r+s}$.

2°. $(a^r)^s = a^{rs}$.

3°. $(ab)^r = a^r \times b^r$.

4°. $\left(\frac{a}{b}\right)^r = \frac{a^r}{b^r}$.

5°. $\frac{a^r}{a^s} = a^{r-s}$,

where $a > 0$, $b > 0$, while r and s are arbitrary rational numbers.

Example 1. Simplify the expression

$$A = (\sqrt{32} + \sqrt{45} - \sqrt{98})(\sqrt{72} - \sqrt{500} - \sqrt{8}).$$

Solution. We first simplify each of the indicated radicals:

$$\sqrt{32} = \sqrt{16 \times 2} = 4\sqrt{2}, \quad \sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5},$$

$$\sqrt{98} = \sqrt{49 \times 2} = 7\sqrt{2},$$

$$\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}, \quad \sqrt{500} = \sqrt{100 \times 5} = 10\sqrt{5},$$

$$\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}.$$

Then the given expression takes the form:

$$A = (4\sqrt{2} + 3\sqrt{5} - 7\sqrt{2})(6\sqrt{2} - 10\sqrt{5} - 2\sqrt{2})$$
$$= (3\sqrt{5} - 3\sqrt{2})(4\sqrt{2} - 10\sqrt{5}).$$

Further, we get:

$$A = 12\sqrt{10} - 24 - 150 + 30\sqrt{10} = 42\sqrt{10} - 174$$
$$= 6(7\sqrt{10} - 29).$$

Example 2. Simplify the expression

$$A=\sqrt{2+\sqrt{3}}\sqrt{2+\sqrt{2+\sqrt{3}}}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$$
$$\times\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}.$$

Solution. We first multiply together the third and fourth multipliers:

$$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$$
$$=\sqrt{4-\left(\sqrt{2+\sqrt{2+\sqrt{3}}}\right)^2}=\sqrt{4-(2+\sqrt{2+\sqrt{3}})}$$
$$=\sqrt{2-\sqrt{2+\sqrt{3}}}.$$

The obtained result is then multiplied by the second multiplier:

$$\sqrt{2-\sqrt{2+\sqrt{3}}}\sqrt{2+\sqrt{2+\sqrt{3}}}=\sqrt{4-(\sqrt{2+\sqrt{3}})^2}$$
$$=\sqrt{4-(2+\sqrt{3})}=\sqrt{2+\sqrt{3}}.$$

This result is finally multiplied by the first multiplier:

$$\sqrt{2-\sqrt{3}}\sqrt{2+\sqrt{3}}=\sqrt{4-(\sqrt{3})^2}=\sqrt{4-3}=1.$$

Thus, $A=1$.

Example 3. Simplify the expression $A=\sqrt[8]{(2-\sqrt{7})^4}$.

Solution. By Property 5°, we get $A=\sqrt{|2-\sqrt{7}|}$. But $2-\sqrt{7}<0$, and therefore $A=\sqrt{-(2-\sqrt{7})}=\sqrt{\sqrt{7}-2}$.

Example 4. Simplify the expression $A=\sqrt{27-10\sqrt{2}}$.

Solution. It is clear that the given expression is simplifiable if it turns out that the radicand is a perfect square of the difference between some two numbers. Let us represent $10\sqrt{2}$ as twice the product of two numbers whose sum of squares is equal to 27, i.e. $10\sqrt{2}=2\sqrt{2}\times 5$.

Thus, $A=\sqrt{2-2\sqrt{2}\times 5+25}=\sqrt{(\sqrt{2}-5)^2}=|\sqrt{2}-5|$, and since $\sqrt{2}-5<0$ we have: $A=5-\sqrt{2}$.

Example 5. Simplify the expression $A=\sqrt[3]{9\sqrt{3}-11\sqrt{2}}$.

Solution. Reasoning as in the preceding example, we write the radicand in the form of a perfect cube of the difference between some two numbers. We have: $9\sqrt{3}=3\sqrt{3}+6\sqrt{3}=(\sqrt{3})^3+3\sqrt{3}(\sqrt{2})^2$

and $11\sqrt{2}=9\sqrt{2}+2\sqrt{2}=3(\sqrt{3})^2\sqrt{2}+(\sqrt{2})^3$.

Thus,

$$A=\sqrt[3]{(\sqrt{3})^3-3(\sqrt{3})^2\sqrt{2}+3\sqrt{3}(\sqrt{2})^2-(\sqrt{2})^3}$$
$$=\sqrt[3]{(\sqrt{3}-\sqrt{2})^3}=\sqrt{3}-\sqrt{2}.$$

Example 6. Rationalize the denominator of the fraction $A=\frac{1}{\sqrt[3]{2}-1}$.

Solution. Multiplying the numerator and denominator of the fraction by the imperfect square of the sum of the numbers $\sqrt[3]{2}$ and 1, we get:

$$A=\frac{(\sqrt[3]{2})^2+\sqrt[3]{2}+1}{(\sqrt[3]{2}-1)((\sqrt[3]{2})^2+\sqrt[3]{2}+1)}=\frac{\sqrt[3]{4}+\sqrt[3]{2}+1}{(\sqrt[3]{2})^3-1^3}=\sqrt[3]{4}+\sqrt[3]{2}+1.$$

Example 7. Rationalize the denominator of the fraction $A=\frac{3}{1+\sqrt{2}-\sqrt{3}}$.

Solution. We first get rid of $\sqrt{3}$ in the denominator. To this end, we multiply both the numerator and denominator by the expression conjugate to the denominator:

$$A=\frac{3(1+\sqrt{2}+\sqrt{3})}{(1+\sqrt{2}-\sqrt{3})(1+\sqrt{2}+\sqrt{3})}=\frac{3(1+\sqrt{2}+\sqrt{3})}{(1+\sqrt{2})^2-3}$$
$$=\frac{3(1+\sqrt{2}+\sqrt{3})}{2\sqrt{2}}.$$

We now get rid of $\sqrt{2}$ in the denominator:

$$A=\frac{3(1+\sqrt{2}+\sqrt{3})\sqrt{2}}{2\sqrt{2}\times\sqrt{2}}=\frac{3(\sqrt{2}+2+\sqrt{6})}{4}.$$

Example 8. Compute the sum $\sqrt[3]{20+\sqrt{392}}+\sqrt[3]{20-\sqrt{392}}$.

Solution. Setting $A=\sqrt[3]{20+\sqrt{392}}+\sqrt[3]{20-\sqrt{392}}$ and cubing both sides of this equality, we get:

$$(20+\sqrt{392})+3(\sqrt[3]{20+\sqrt{392}})^2\sqrt[3]{20-\sqrt{392}}$$
$$+3\sqrt[3]{20+\sqrt{392}}(\sqrt[3]{20-\sqrt{392}})^2+(20-\sqrt{392})=A^3,$$

or

$$40+3\sqrt[3]{20+\sqrt{392}}\sqrt[3]{20-\sqrt{392}}(\sqrt[3]{20+\sqrt{392}}$$
$$+\sqrt[3]{20-\sqrt{392}}=A^3,$$

where

$$\sqrt[3]{20+\sqrt{392}}+\sqrt[3]{20-\sqrt{392}}=A.$$

Thus, we get: $40+3\sqrt[3]{20^2-(\sqrt{392})^2}\cdot A=A^3$, $40+6A=A^3$, $A^3-6A-40=0$.

But $A^3-6A-40=(A^3-4A^2)+(4A^2-16A)+(10A-40)=A^2(A-4)+4A(A-4)+10(A-4)=(A-4)\times(A^2+4A+10)$.

Since $A^2+4A+10=(A^2+4A+4)+6=(A+2)^2+6\neq 0$, the equality $(A-4)(A^2+4A+10)=0$ is fulfilled only for $A=4$.

Thus, $\sqrt[3]{20+\sqrt{392}}+\sqrt[3]{20-\sqrt{392}}=4$.

Example 9. Transform the function

$$f(a)=\sqrt{a^2-4a+4}+\sqrt{a^2+6a+9}$$

to the form containing no radical and modulus signs.

Solution. Since $\sqrt{a^2-4a+4}=\sqrt{(a-2)^2}=|a-2|$ and $\sqrt{a^2+6a+9}=\sqrt{(a+3)^2}=|a+3|$, we have: $f(a)=|a-2|+|a+3|$.

The points $a_1=-3$ and $a_2=2$ divide the number line into the three intervals: $(-\infty,-3)$, $[-3,2)$, and $[2,\infty)$. Consider the given function on each of these intervals.

For $a<-3$ we have: $|a-2|=-a+2$, $|a+3|=-a-3$, i.e. $f(a)=-a+2-a-3=-2a-1$.

For $-3\leqslant a<2$ we have: $|a-2|=-a+2$, $|a+3|=a+3$, and then $f(a)=-a+2+a+3=5$.

Finally, for $a\geqslant 2$ we have: $|a-2|=a-2$, $|a+3|=a+3$, that is, $f(a)=a-2+a+3=2a+1$.

Thus, $f(a)=\begin{cases}-2a-1 & \text{if } a<-3,\\ 5 & \text{if } -3\leqslant a<2,\\ 2a+1 & \text{if } a\geqslant 2.\end{cases}$

Example 10. Simplify the function

$$f(a,b)=\sqrt{\frac{a+b^2}{b}+2\sqrt{a}}-\sqrt{\frac{a+b^2}{b}-2\sqrt{a}},$$

where $a\geqslant 0$, $b>0$.

Solution. $f(a,b)=\sqrt{\frac{(\sqrt{a}+b)^2}{b}}-\sqrt{\frac{(\sqrt{a}-b)^2}{b}}=$
$\frac{|\sqrt{a}+b|-|\sqrt{a}-b|}{\sqrt{b}}$.

Since $a \geqslant 0$, $b > 0$, we have: $\sqrt{a}+b > 0$, and, consequently, $|\sqrt{a}+b| = \sqrt{a}+b$. Hence,

$$f(a, b) = \frac{\sqrt{a}+b-|\sqrt{a}-b|}{\sqrt{b}}.$$

Now, we have to consider two cases: (1) $\sqrt{a}-b \geqslant 0$, (2) $\sqrt{a}-b < 0$.

In the first case we have: $|\sqrt{a}-b| = \sqrt{a}-b$, and, consequently

$$f(a, b) = \frac{\sqrt{a}+b-\sqrt{a}+b}{\sqrt{b}} = 2\sqrt{b}.$$

In the second case: $|\sqrt{a}-b| = -(\sqrt{a}-b)$, and, consequently,

$$f(a, b) = \frac{\sqrt{a}+b+\sqrt{a}-b}{\sqrt{b}} = \frac{2\sqrt{ab}}{b}.$$

Thus, $f(a, b) = \begin{cases} 2\sqrt{b} & \text{if } \sqrt{a} \geqslant b, \\ \dfrac{2\sqrt{ab}}{b} & \text{if } \sqrt{a} < b. \end{cases}$

EXERCISES

In Problems 120 through 125, evaluate the indicated functions:

120. $2a^2-5ab+2b^2$ for $a=\sqrt{6}+\sqrt{5}$ and $b=\sqrt{6}-\sqrt{5}$.

121. $3a^2+4ab-3b^2$ for $a=\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ and $b=\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$.

122. $4a^3+2a^2-8a+7$ for $a=\dfrac{1}{2}(\sqrt{3}+1)$.

123. $\dfrac{a+b-1}{a-b+1}$ for $a=\dfrac{\sqrt{x}+1}{\sqrt{xy}+1}$ and $b=\dfrac{\sqrt{xy}+\sqrt{x}}{\sqrt{xy}-1}$.

124. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$ for $x=\dfrac{2ab}{1+b^2}$.

125. $2a(1+x^2)^{\frac{1}{2}}(x+(1+x^2)^{\frac{1}{2}})^{-1}$ for $x=\dfrac{1}{2}\left((ab^{-1})^{\frac{1}{2}}-(ba^{-1})^{\frac{1}{2}}\right)$.

In Problems 126 through 134, simplify the given functions:

126. $\sqrt{7+4\sqrt{3}}$. 127. $\sqrt{3-2\sqrt{2}}$.

128. $\left(\sqrt{5+2\sqrt{6}}+\sqrt{5-2\sqrt{6}}\right)\frac{\sqrt{3}}{2}$.

129. $\frac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\frac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$. 130. $\sqrt{4\sqrt{2}+2\sqrt{6}}$.

131. $\sqrt[4]{17+\sqrt{288}}$. 132. $\sqrt[4]{28-16\sqrt{3}}$.

133. $\sqrt{17-4\sqrt{9+4\sqrt{5}}}$. 134. $\sqrt{3+\sqrt{5-\sqrt{13+\sqrt{48}}}}$.

In Problems 135 through 143, rationalize the denominator of the given fraction:

135. $\frac{1}{\sqrt[4]{5}-\sqrt[4]{2}}$. 136. $\frac{1}{\sqrt[3]{15}-\sqrt[3]{7}}$.

137. $\frac{\sqrt{\sqrt{5}+\sqrt{3}}}{\sqrt{\sqrt{5}-\sqrt{3}}}$. 138. $\frac{1}{1+\sqrt{2}+\sqrt{3}}$.

139. $\frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}$. 140. $\frac{1}{\sqrt{14}+\sqrt{21}+\sqrt{15}+\sqrt{10}}$.

141. $\frac{1}{\sqrt[4]{2}+\sqrt[4]{4}+\sqrt[4]{8}+2}$. 142. $\frac{1}{\sqrt{\sqrt{2}+\sqrt[3]{3}}}$.

143. $\frac{2+\sqrt{6}}{2\sqrt{2}+2\sqrt{3}-\sqrt{6}-2}$.

In Problems 144 through 151, check whether the given equalities are true:

144. $$\frac{\sqrt{\sqrt[4]{8}-\sqrt{\sqrt{2}+1}}}{\sqrt{\sqrt[4]{8}+\sqrt{\sqrt{2}-1}}-\sqrt{\sqrt[4]{8}-\sqrt{\sqrt{2}-1}}}=\frac{1}{\sqrt{2}}.$$

145. $\sqrt[3]{26+15\sqrt{3}}\,(2-\sqrt{3})=1$. 146. $\frac{2\sqrt[3]{2}}{1+\sqrt{3}}=\frac{\sqrt[3]{20+12\sqrt{3}}}{2+\sqrt{3}}$.

147. $$\frac{\sqrt{5-2\sqrt{6}}\,(5+2\sqrt{6})\,(49-20\sqrt{6})}{\sqrt{27}-3\sqrt{18}+3\sqrt{12}-\sqrt{8}}=1.$$

148. $$\left(\frac{6+4\sqrt{2}}{\sqrt{2}+\sqrt{6+4\sqrt{2}}}+\frac{6-4\sqrt{2}}{\sqrt{2}-\sqrt{6-4\sqrt{2}}}\right)^2=8.$$

149. $$\left(\frac{3}{\sqrt[3]{64}-\sqrt[3]{25}}+\frac{\sqrt[3]{40}}{\sqrt[3]{8}+\sqrt[3]{5}}-\frac{10}{\sqrt[3]{25}}\right)^{-1}(13-4\sqrt[3]{5}-2\sqrt[3]{25})+\sqrt[3]{25}=4.$$

150. $$\sqrt[3]{6+\sqrt{\frac{847}{27}}}+\sqrt[3]{6-\sqrt{\frac{847}{27}}}=3.$$

151. $\sqrt[3]{5\sqrt{2}+7}-\sqrt[3]{5\sqrt{2}-7}=2$.

In Problems 152 through 156, prove the given identities and indicate the domain of definition:

152. $\dfrac{2a^{-\frac{1}{3}}}{a^{\frac{2}{3}}-3a^{\frac{1}{3}}}-\dfrac{a^{\frac{2}{3}}}{a^{\frac{5}{3}}-a^{\frac{2}{3}}}-\dfrac{a+1}{a^2-4a+3}=0.$

153. $\sqrt[4]{6a(5+2\sqrt{6})}\sqrt{3\sqrt{2a}-2\sqrt{3a}}=\sqrt{6a}.$

154. $\dfrac{\sqrt[3]{\sqrt{3}-\sqrt{5}}\,\sqrt[6]{8+2\sqrt{15}}-\sqrt[3]{a}}{\sqrt[3]{\sqrt{20}+\sqrt{12}}\,\sqrt[6]{8-2\sqrt{15}}-2\sqrt[3]{2a}+\sqrt[3]{a^2}}=\dfrac{\sqrt[3]{a^2}+\sqrt[3]{2a}+\sqrt[3]{4}}{2-a}.$

155. $\dfrac{((\sqrt[4]{a}+\sqrt[4]{b})^2-(\sqrt[4]{a}-\sqrt[4]{b})^2)^2-(16a+4b)}{4a-b}+\dfrac{10\sqrt{a}-3\sqrt{b}}{2\sqrt{a}+\sqrt{b}}=1.$

156. $\sqrt{\left(a^2+\dfrac{4}{a^2}\right)^2-8\left(a+\dfrac{2}{a}\right)^2+48}=\left(a-\dfrac{2}{a}\right)^2.$

In Problems 157 through 159, prove the given identities:

157. $\dfrac{a^2+2a-3+(a+1)\sqrt{a^2-9}}{a^2-2a-3+(a-1)\sqrt{a^2-9}}=\dfrac{\sqrt{a+3}}{\sqrt{a-3}}$ if $a>3$.

158. $\sqrt{\sqrt{a}+\sqrt{\dfrac{a^2-4}{a}}}+\sqrt{\sqrt{a}-\sqrt{\dfrac{a^2-4}{a}}}=\dfrac{\sqrt{2a+4}}{\sqrt[4]{a}}$ if $a\geqslant 2$.

159. $\left(\sqrt[3]{(x^2+1)\sqrt{1+\dfrac{1}{x^2}}}+\sqrt[3]{(x^2-1)\sqrt{1-\dfrac{1}{x^2}}}\right)^{-2}=$
$\dfrac{\sqrt[3]{x^2}\,(x^2-\sqrt{x^4-1})}{2}$ if $x>1$.

In Problems 160 through 181, simplify the given expressions:

160. $(0.5a^{0.25}+a^{0.75})^2-a^{1.5}(1+a^{-0.5}).$

161. $\left(\dfrac{\sqrt[4]{ab}-\sqrt{ab}}{1-\sqrt{ab}}+\dfrac{1-\sqrt[4]{ab}}{\sqrt[4]{ab}}\right)\div\dfrac{\sqrt[4]{ab}}{1+\sqrt[4]{a^3b^3}}-\dfrac{1-\sqrt[4]{ab}-\sqrt{ab}}{\sqrt{ab}}.$

162. $\dfrac{m+n}{\sqrt{m}+\sqrt{n}}\div\left(\dfrac{m+n}{\sqrt{mn}}+\dfrac{n}{m-\sqrt{mn}}-\dfrac{m}{\sqrt{mn}+n}\right).$

163. $\left(\dfrac{a\sqrt[3]{a}-2a\sqrt[3]{b}+\sqrt[3]{a^2b^2}}{\sqrt[3]{a^2}-\sqrt[3]{ab}}+\dfrac{\sqrt[3]{a^2b}-\sqrt[3]{ab^2}}{\sqrt[3]{a}-\sqrt[3]{b}}\right)\div\sqrt[3]{a}.$

164. $\left(\dfrac{1}{\sqrt{a}+\sqrt{a+1}}+\dfrac{1}{\sqrt{a}-\sqrt{a-1}}\right)\div\left(1+\sqrt{\dfrac{a+1}{a-1}}\right).$

165. $\left(\dfrac{\sqrt{1+a}}{\sqrt{1+a}-\sqrt{1-a}}+\dfrac{1-a}{\sqrt{1-a^2}-1+a}\right)\left(\sqrt{\dfrac{1}{a^2}-1}-\dfrac{1}{a}\right).$

166. $\left(m+\frac{n^{1.5}}{m^{0.5}}\right)^{\frac{2}{3}}\left(\frac{m^{0.5}-n^{0.5}}{m^{0.5}}+\frac{n^{0.5}}{m^{0.5}-n^{0.5}}\right)^{-\frac{2}{3}}$.

167. $2a\sqrt{1+0.25\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2}\div\left(0.5\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)+\right.$
$\left.\sqrt{1+0.25\left(\sqrt{\frac{a}{b}}-\sqrt{\frac{b}{a}}\right)^2}\right)$, where $a>0$, $b>0$.

168. $\left(\frac{1}{\sqrt{a}-4\sqrt{a^{-1}}}-\frac{2\sqrt[3]{a}}{\sqrt[3]{a^4}-\sqrt[3]{64a}}\right)^{-2}-\sqrt{a^2+8a+16}$.

169. $\left(\sqrt{\left(\frac{a+b}{2\sqrt{ab}}-1\right)^{-1}}+\sqrt{\left(\frac{a+b}{2\sqrt{ab}}+1\right)^{-1}}\right)\div$
$\left(\sqrt{\left(\frac{a+b}{2\sqrt{ab}}-1\right)^{-1}}-\sqrt{\left(\frac{a+b}{2\sqrt{ab}}+1\right)^{-1}}\right)$, where $a>0$, $b>0$.

170. $\left(\sqrt{\frac{(1-a)\sqrt[3]{1+a}}{a}}\sqrt[3]{\frac{3a^2}{4-8a+4a^2}}\right)^{-1}-\sqrt[3]{\left(\frac{3a\sqrt{a}}{2\sqrt{1-a^2}}\right)^{-1}}$.

171. $\left(\frac{(1-a)^{\frac{1}{4}}}{2(1+a)^{\frac{3}{4}}}+\frac{(1+a)^{\frac{1}{4}}(1-a)^{-\frac{3}{4}}}{2}\right)(1-a)^{-\frac{1}{2}}\left(\frac{1+a}{1-a}\right)^{-\frac{1}{4}}$.

172. $\left(\frac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}+\frac{1-\frac{a}{\sqrt{a^2-1}}}{1+\frac{a}{\sqrt{a^2-1}}}\right)\div\frac{\sqrt{a-\frac{1}{a}}}{\sqrt{\frac{1}{a}}}$.

173. $b\left(\left(\frac{a\sqrt[4]{a}+\sqrt[4]{a^2b^3}}{\sqrt[4]{a^3}+\sqrt[4]{a^2b}}-\sqrt[4]{ab}\right)\div(\sqrt[4]{a}-\sqrt[4]{b})-\sqrt[4]{a}\right)^{-4}$.

174. $\left(\frac{\sqrt[3]{a^2b}-\sqrt[3]{ab^2}}{\sqrt[3]{a^2}-2\sqrt[3]{ab}+\sqrt[3]{b^2}}-\frac{a+b}{\sqrt[3]{a^2}-\sqrt[3]{b^2}}\right)\left(\sqrt[6]{a}-\sqrt[6]{b}\right)^{-1}+\sqrt[6]{a}$.

175. $\left(\frac{1}{a^{\frac{1}{3}}-a^{\frac{1}{6}}+1}+\frac{1}{a^{\frac{1}{3}}+a^{\frac{1}{6}}+1}-\frac{2a^{\frac{1}{3}}-2}{a^{\frac{2}{3}}-a^{\frac{1}{3}}+1}\right)^{-1}-\frac{1}{4}a^{\frac{4}{3}}$.

176. $\left(\frac{\sqrt[4]{b}\,(\sqrt[4]{a}-\sqrt[4]{b})+2\sqrt[4]{ab}}{(\sqrt[4]{b}+\sqrt[4]{a})^2}-\left(\sqrt[4]{\frac{b}{a}}+1\right)^{-1}+1\right)^{\frac{1}{2}}\sqrt[8]{ab}$.

177. $\left(\frac{(a+b)\left(a^{\frac{2}{3}}-b^{\frac{2}{3}}\right)^{-1}-(\sqrt[3]{a^2b}-\sqrt[3]{ab^2})\left(b^{\frac{1}{3}}-a^{\frac{1}{3}}\right)^{-2}}{(\sqrt[6]{a}+\sqrt[6]{b})(\sqrt[3]{b}+\sqrt[6]{ab}-2\sqrt[3]{a})}\right)^{-1}+2\sqrt[6]{a}$.

178. $(\sqrt{ab}-ab\,(a+\sqrt{ab})^{-1})\div\frac{2\sqrt{ab}-2b}{a-b}$.

179. $\left(\frac{(a-1)^{-1}}{a^{-3}}-(1-a)^{-1}\right)\left(\frac{1+a(a-2)}{a^2-a+1}\right)\sqrt{\frac{1}{(a+1)^2}}$.

180. $\left(\frac{4b^2+2ab}{\sqrt{4a^2b^2-8ab^3}}-\frac{16^{\frac{3}{4}}b^{\frac{3}{2}}}{\sqrt{4a^2b-8ab^2}}\right)\left(\frac{1}{2ab}-a^{-2}\right)^{-\frac{1}{2}}\sqrt{\frac{2a}{b}}$.

181. $\frac{\sqrt[6]{b^5}-\sqrt[6]{a^2b^3}+\sqrt[6]{a^3b^2}-\sqrt[6]{a^5}}{\sqrt[6]{b}+\sqrt[6]{a}}\left(\frac{\sqrt[6]{ab^9}+\sqrt[6]{a^{10}}}{a-\sqrt{ab}+b}\right)^{-1}+1$.

SEC. 4. IDENTICAL TRANSFORMATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Let us recall the fundamentals of logarithms which are needed for solving the problems contained in this section.

Let a be a positive number different from 1. The number x is defined as the *logarithm of the number N to the base a* if $a^x = N$.

For instance, $\log_2 16 = 4$, since $2^4 = 16$, $\log_{\sqrt{3}}\frac{1}{81} = -8$, since $(\sqrt{3})^{-8} = \frac{1}{81}$. In general, $\log_a a^r = r$.

The definition of logarithm implies that, firstly, the two notations $x = \log_a N$ and $a^x = N$ have the same meaning; secondly, the number N must be positive; thirdly, if $a > 0$, $a \neq 1$, $N > 0$, then

$$a^{\log_a N} = N. \qquad (1)$$

Identity (1) is actually a mathematic notation of the definition of logarithm; it is also called the *fundamental logarithmic identity*.

For any positive number N and any positive number a different from 1 there exists only one real number x such that $x = \log_a N$. Hence it follows, particularly, that if $N_1 = N_2$, then $\log_a N_1 = \log_a N_2$, where $N_1 > 0$, $N_2 > 0$.

Let us recall the basic properties of logarithms:

If $N_1 \cdot N_2 > 0$, then

1°. $\log_a (N_1 \cdot N_2) = \log_a |N_1| + \log_a |N_2|$.

2°. $\log_a (N_1/N_2) = \log_a |N_1| - \log_a |N_2|$.

If, in particular, $N_1 > 0$, $N_2 > 0$, then $|N_1| = N_1$, $|N_2| = N_2$ and we get: $\log_a (N_1 \cdot N_2) = \log_a N_1 + \log_a N_2$, $\log_a (N_1/N_2) = \log_a N_1 - \log_a N_2$.

3°. If $N > 0$, $\mu \in R$, then $\log_a N^\mu = \mu \log_a N$; if $N \neq 0$, $\mu = 2m$ $(m = \pm 1, \pm 2, \ldots)$, then $\log_a N^\mu = \mu \log_a |N|$.

4°. If $N > 0$, $b > 0$, $b \neq 1$, then $\log_a N = \log_b N/\log_b a$. This identity is customarily called the *formula for change of base*. For $N = b$, in particular, it implies that $\log_a b = 1/\log_b a$.

5°. If $N > 0$, $\mu \in R$, then $\log_a N = \log_{a^\mu} N^\mu$.

Consider several examples.

Example 1. Compute $49^{1-0.25\log_7 25}$.

Solution. Since $49 = 7^2$ and the exponents are multiplied when raising a power to a power, we get:

$$7^{2-0.5\log_7 25}.$$

The exponent can be transformed in the following way:

$$2-0.5\log_7 25 = 2-\log_7 5 = \log_7 49 - \log_7 5 = \log_7 \frac{49}{5}.$$

Thus, $7^{2-0.5\log_7 25} = 7^{\log_7 \frac{49}{5}}$. But Identity (1) implies that $7^{\log_7 \frac{49}{5}} = \frac{49}{5}$. Thus, $49^{1-0.25\log_7 25} = 9.8$.

Example 2. Compute $\log 25$ if $\log 2 = a$.

Solution. We have $\log 25 = 2\log 5$. Let us now express the number 5 in terms of the numbers 10 and 2 (that is, in terms of the given base and the number whose logarithm is known), using the operations of multiplication, division, and involution (raising to a power). Since $5 = \frac{10}{2}$, we have $2\log 5 = 2\log \frac{10}{2} = 2(\log 10 - \log 2) = 2(1-a)$.

Example 3. Compute $\log_3 18$ if $\log_3 12 = a$.

Solution. First Method. The same as in the preceding example, we simplify $\log_3 18$:

$$\log_3 18 = \log_3 (3^2 \times 2) = 2 + \log_3 2.$$

Hence, we have to compute $\log_3 2$ knowing that $\log_3 12 = a$. Let us express the number 2 in terms of the numbers 3 (the given base) and 12 (the number whose logarithm is known), using the operations of multiplication, division, and involution.

We have: $2 = \sqrt{\frac{12}{3}}$, but then

$$\log_3 2 = \log_3 \sqrt{\frac{12}{3}} = 0.5(\log_3 12 - \log_3 3) = 0.5(a-1).$$

Thus, $\log_3 18 = 2 + \frac{a-1}{2} = \frac{a+3}{2}$.

Second Method. We have: $\log_3 18 = 2 + \log_3 2$. Introducing the notation $\log_3 2 = x$, we get $\log_3 18 = 2 + x$.

Further, $\log_3 12 = \log_3 (3 \times 2^2) = 1 + 2\log_3 2 = 1 + 2x$.

But, by hypothesis, $\log_3 12 = a$, consequently, $1 + 2x = a$, whence $x = \frac{a-1}{2}$.

Thus, $\log_3 18 = 2 + x = 2 + \frac{a-1}{2} = \frac{a+3}{2}$.

Example 4. Compute $\log_{49} 16$ if $\log_{14} 28 = a$.

Solution. Applying Formulas 5° and 3°, we get:

$$\log_{49} 16 = \log_{\sqrt{49}} \sqrt{16} = \log_7 4 = 2 \log_7 2.$$

Setting $\log_7 2 = x$, we have: $\log_{49} 16 = 2x$. Further, we have:

$$\log_{14} 28 = \frac{\log_7 28}{\log_7 14} = \frac{\log_7 (2^2 \times 7)}{\log_7 (2 \times 7)} = \frac{2 \log_7 2 + \log_7 7}{\log_7 2 + \log_7 7} = \frac{2x+1}{x+1}.$$

Since, by hypothesis, $\log_{14} 28 = a$, the problem is reduced to solving the equation $\frac{2x+1}{x+1} = a$, whence we find: $x = \frac{a-1}{2-a}$.

Thus, $\log_{49} 16 = 2x = \frac{2(a-1)}{2-a}$.

Example 5. Compute $\log_{12} 60$ if $\log_6 30 = a$, $\log_{15} 24 = b$.

Solution.

$$\log_{12} 60 = \frac{\log_2 60}{\log_2 12} = \frac{\log_2 (4 \times 3 \times 5)}{\log_2 (4 \times 3)} = \frac{2 + \log_2 3 + \log_2 5}{2 + \log_2 3}.$$

Setting $\log_2 3 = x$, $\log_2 5 = y$, we get: $\log_{12} 60 = \frac{2+x+y}{2+x}$. Further, we have:

$$a = \log_6 30 = \frac{\log_2 30}{\log_2 6} = \frac{\log_2 (2 \times 3 \times 5)}{\log_2 (2 \times 3)} = \frac{1+x+y}{1+x},$$

$$b = \log_{15} 24 = \frac{\log_2 24}{\log_2 15} = \frac{\log_2 (8 \times 3)}{\log_2 (3 \times 5)} = \frac{3+x}{x+y}.$$

Thus, the problem is reduced to solving the following system of equations: $\begin{cases} \frac{1+x+y}{1+x} = a, \\ \frac{x+3}{x+y} = b. \end{cases}$

From this system we find:

$$x = \frac{b+3-ab}{ab-1}, \quad y = \frac{2a-b-2+ab}{ab-1}.$$

Then $\log_{12} 60 = \frac{2ab+2a-1}{ab+b+1}$.

EXERCISES

In Problems 182 through 187, compute the given expressions:

182. (a) $-\log_8 \log_4 \log_2 16$; (b) $-\log_2 \log_3 \sqrt{\sqrt[4]{3}}$; (c) $\log \log \sqrt[2]{\sqrt[5]{10}}$.

183. (a) $\left(\frac{16}{15}\right)^{\log_{\frac{125}{64}} 3}$; (b) $\left(\frac{8}{27}\right)^{\log_{\frac{81}{16}} 5}$.

184. (a) $36^{\log_6 5}+10^{1-\log 2}-3^{\log_9 36}$; (b) $81^{1/\log_5 3}+27^{\log_9 36}+3^{4/\log_7 9}$.

185. $\log\left(2-\log_{\frac{1}{3}}\sqrt{3}\log_{\sqrt{3}}\frac{1}{3}\right)$.

186. (a) $\log_3 7\log_7 5\log_5 4+1$; (b) $\log_3 2\log_4 3\log_5 4\log_6 5\log_7 6\log_8 7$.

187. (a) $2^{\log_3 5}-5^{\log_3 2}$; (b) $3^{\sqrt{\log_3 2}}-2^{\sqrt{\log_2 3}}$.

In Problems 188 through 199, compute the indicated expressions:

188. $\log 1250$ if $\log 2 = 0.3010$.
189. $\log_{100} 40$ if $\log_2 5 = a$.
190. $\log_6 16$ if $\log_{12} 27 = a$.
191. $\log_3 5$ if $\log_6 2 = a$, $\log_6 5 = b$.
192. $\log_{35} 28$ if $\log_{14} 7 = a$, $\log_{14} 5 = b$.
193. $\log_{\sqrt{3}}\sqrt[6]{a}$ if $\log_a 27 = b$, $a > 0$, $a \neq 1$.
194. $\log_5 3.38$ if $\log 2 = a$, $\log 13 = b$.
195. $\log_2 360$ if $\log_3 20 = a$, $\log_3 15 = b$.
196. $\log_{275} 60$ if $\log_{12} 5 = a$, $\log_{12} 11 = b$.
197. $\log_c ab$ if $\log_a n = p$, $\log_b n = q$, $\log_c n = r$, where a, b, c, n are positive numbers different from 1.

198. $\log_{ab}\dfrac{\sqrt[3]{a}}{\sqrt{b}}$ if $\log_{ab} a = n$, where a, b are positive numbers and $ab \neq 1$.

199. $\log_{abc} n$ if $\log_a n=2$, $\log_b n=3$, $\log_c n=6$, where a, b, c are positive numbers different from 1.

In Problems 200 through 206, prove the given identities:

200. $b^{\log_a c}=c^{\log_a b}$.

201. (a) $\log_{ab} n=\dfrac{\log_a n\log_b n}{\log_a n+\log_b n}$; (b) $\dfrac{\log_a n}{\log_{ab} n}=1+\log_a b$;

(c) $\log_{bn} an=\dfrac{\log_b a+\log_b n}{1+\log_b n}$.

202. $\dfrac{1}{\log_a n}+\dfrac{1}{\log_{a^2} n}+\dfrac{1}{\log_{a^3} n}+\dfrac{1}{\log_{a^4} n}+\dfrac{1}{\log_{a^5} n}=15\log_n a$.

203. $\dfrac{1}{(\log_{a_1} x)^{-1}+(\log_{a_2} x)^{-1}+\ldots+(\log_{a_n} x)^{-1}}=\log_{a_1 a_2 \ldots a_n} x$.

204. $\log_a n\,\log_b n+\log_b n\,\log_c n+\log_c n\,\log_a n=\dfrac{\log_a n\,\log_b n\,\log_c n}{\log_{abc} n}$.

205. $\log\dfrac{a+b}{3}=\dfrac{1}{2}(\log a+\log b)$ if $a^2+b^2=7ab$.

206. $\log\dfrac{a+2b}{4}=\dfrac{1}{2}(\log a+\log b)$ if $a^2+4b^2=12ab$.

In Problems 207 through 215, simplify the given expressions;

207. $(\log_a b+\log_b a+2)(\log_a b-\log_{ab} b)\log_b a-1$.

208. $\dfrac{1-\log_a^3 b}{(\log_a b+\log_b a+1)\log_a\frac{a}{b}}$.

209. $\left(b^{\frac{\log_{100} a}{\log a}} a^{\frac{\log_{100} b}{\log b}}\right)^{2 \log_{ab}(a+b)}$. **210.** $0.2\left(2a^{\log_2 b}+3b^{\log_{\sqrt{2}} \sqrt{a}}\right)$.

211. $\sqrt{1+2^{\frac{\log a}{\log \sqrt{2}}}-a^{1+\frac{1}{\log_4 a^2}}}-1$.

212. $\sqrt{\log_a b+\log_b a+2}\cdot\log_{ab} a\cdot\sqrt{\log_a^3 b}$.

213. $\sqrt{\sqrt{\log_b^4 a+\log_a^4 b+2}+2}-\log_b a-\log_a b$.

214. $\dfrac{\log_a b-\log_{\frac{\sqrt{a}}{b^3}} \sqrt{b}}{\log_{\frac{a}{b^4}} b-\log_{\frac{a}{b^6}} b} \div \log_b (a^3 b^{-12})$.

215. $2 \log_a^{\frac{1}{2}} b\left((\log_a \sqrt[4]{ab}+\log_b \sqrt[4]{ab})^{\frac{1}{2}}-\left(\log_a \sqrt[4]{\frac{b}{a}}+\log_b \sqrt[4]{\frac{a}{b}}\right)^{\frac{1}{2}}\right)$
if $a>1$, $b>1$.

SEC. 5. PROVING INEQUALITIES

The present section deals with inequalities whose validity is required to be ascertained on a given set of values of constituent letters. If such a set is not indicated, then it is implied that the letters may be any real values.

1. Proving Inequalities with the Aid of Definition. As is known, by definition, $a>b$ if $a-b$ is a positive number. Therefore, to prove the inequality $f(a, b, \ldots, k)>q(a, b, \ldots, k)$ on a given set of values of letters $a, b, \ldots, k$, we form the difference $f(a, b, \ldots, k)-q(a, b, \ldots, k)$ and prove that it is positive for the given values of $a, b, \ldots, k$ (similarly, this technique is used for proving inequalities of the form $f<q$, $f \geqslant q$, $f \leqslant q$).

Example 1. Prove that if $a \geqslant 0$, $b \geqslant 0$, then

$$\frac{a+b}{2} \geqslant \sqrt{ab} \text{ (Cauchy's inequality).} \qquad (1)$$

Proof. Let us form the difference $\frac{a+b}{2}-\sqrt{ab}$ and determine its sign. We have: $\frac{a+b}{2}-\sqrt{ab}=\frac{a-2\sqrt{ab}+b}{2}=\frac{(\sqrt{a}-\sqrt{b})^2}{2}$.

For any nonnegative values of a and b the expression $\frac{(\sqrt{a}-\sqrt{b})^2}{2}$ is nonnegative. It becomes equal to zero if and only if $a=b$. Thus, the difference $\frac{a+b}{2}-\sqrt{ab}$ is nonnegative, and this means that $\frac{a+b}{2} \geqslant \sqrt{ab}$. The equality sign takes place only for $a=b$.

Example 2. Prove that if $ab > 0$, then

$$\frac{a}{b}+\frac{b}{a} \geqslant 2. \tag{2}$$

Proof. We have:

$$\left(\frac{a}{b}+\frac{b}{a}\right)-2=\frac{a^2+b^2-2ab}{ab}=\frac{(a-b)^2}{ab}.$$

Since $ab > 0$, we have: $\frac{(a-b)^2}{ab} \geqslant 0$, the equality sign taking place only for $a = b$. Thus, the difference $\left(\frac{a}{b}+\frac{b}{a}\right)-2$ is nonnegative, that is, Inequality (2) has been proved.

Example 3. Prove that

$$a^2 + 4b^2 + 3c^2 + 14 > 2a + 12b + 6c. \tag{3}$$

Proof. Consider the difference

$$(a^2 + 4b^2 + 3c^2 + 14) - (2a + 12b + 6c).$$

Rearranging the terms of this difference, we get:

$$(a^2 - 2a + 1) + (4b^2 - 12b + 9) + (3c^2 - 6c + 3) + 1$$
$$= (a - 1)^2 + (2b - 3)^2 + 3(c - 1)^2 + 1.$$

The last expression is positive for any values of a, b, c.

Inequality (3) has been proved.

Example 4. Prove that if $a + b + c \geqslant 0$, then

$$a^3 + b^3 + c^3 \geqslant 3abc. \tag{4}$$

Proof. Consider the difference $a^3 + b^3 + c^3 - 3abc$, in which the sum $a^3 + b^3$ is completed to the cube of a sum. We get:

$$a^3 + b^3 + c^3 - 3abc = a^3 + 3a^2b + 3ab^2 + b^3 + c^3 - 3a^2b$$
$$-3ab^2 - 3abc = (a + b)^3 - 3ab(a + b + c) + c^3.$$

Factoring the sum of cubes $(a + b)^3 + c^3$, we get:

$$(a + b)^3 + c^3 - 3ab(a + b + c) = ((a + b) + c)((a + b)^2$$
$$- (a + b)c + c^2) - 3ab(a + b + c) = (a + b + c)(a^2 + 2ab + b^2$$
$$- ac - bc + c^2 - 3ab) = (a + b + c)(a^2 + b^2 + c^2 - ab - bc$$
$$- ac) = 0.5(a + b + c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac)$$
$$= 0.5(a + b + c)((a - b)^2 + (a - c)^2 + (b - c)^2).$$

Since, by hypothesis, $a + b + c \geqslant 0$, the obtained expression is nonnegative. Hence it follows that Inequality (4) is true. Note that the equality sign occurs in Inequality (4) if $a + b + c = 0$ and also if $a = b = c$.

2. Synthetic Method of Proving Inequalities. This method consists in that, with the aid of some transformations, the inequality to be proved is derived from some known (reference) inequalities. For instance, the following inequalities may serve as reference ones: (a) $a^2 \geqslant 0$; (b) $\frac{a+b}{2} \geqslant \sqrt{ab}$, where $a \geqslant 0$, $b \geqslant 0$; (c) $\frac{a}{b} + \frac{b}{a} \geqslant 0$, where $ab > 0$; (d) $ax^2 + bx + c > 0$, where $a > 0$ and $b^2 - 4ac < 0$.

Example 5. Prove that if $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$, $d \geqslant 0$, then

$$\frac{a+b+c+d}{4} \geqslant \sqrt[4]{abcd}. \tag{5}$$

Proof. Let us take Cauchy's inequality as a reference one:

$$\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geqslant \sqrt{\frac{a+b}{2} \frac{c+d}{2}}.$$

Since, in turn, $\frac{a+b}{2} \geqslant \sqrt{ab}$ and $\frac{c+d}{2} \geqslant \sqrt{cd}$, we have:

$$\sqrt{\frac{a+b}{2} \frac{c+d}{2}} \geqslant \sqrt{\sqrt{ab}\sqrt{cd}} = \sqrt[4]{abcd}.$$

Hence, $\dfrac{\frac{a+b}{2} + \frac{c+d}{2}}{2} \geqslant \sqrt[4]{abcd}.$

But $\dfrac{\frac{a+b}{2} + \frac{c+d}{2}}{2} = \dfrac{a+b+c+d}{4}.$

Thus, $\frac{a+b+c+d}{4} \geqslant \sqrt[4]{abcd}.$

On having analysed the proof, we arrive at the conclusion that the equality sign occurs in Inequality (5) if and only if $a = b$, $c = d$ and $\frac{a+b}{2} = \frac{c+d}{2}$, that is, if $a = b = c = d$.

Example 6. Prove that $\left(\frac{a+1}{2}\right)^n > n!$, where $n \in N$, $n > 1$.

Proof. Let us take as reference inequalities the following Cauchy's inequalities:

$$\frac{n+1}{2} \geqslant \sqrt{n \times 1}; \quad \frac{(n-1)+2}{2} \geqslant \sqrt{(n-1) \times 2};$$

$$\frac{(n-2)+3}{2} \geqslant \sqrt{(n-2) \times 3}, \ \ldots;$$

$$\frac{2+(n-1)}{2} \geqslant \sqrt{2 \times (n-1)}; \quad \frac{1+n}{2} \geqslant \sqrt{1 \times n}.$$

Multiplying together these n inequalities, we get:

$$\left(\frac{n+1}{2}\right)^n \geqslant \sqrt{(n(n-1)(n-2)\ \ldots\ 2\times 1)(1\times 2\times 3\ \ldots\ (n-1)n)}$$
$$=\sqrt{n!n!}=\sqrt{(n!)^2}=n!.$$

Thus,

$$\left(\frac{n+1}{2}\right)^n \geqslant n!. \tag{6}$$

Since, by hypothesis, $n \neq 1$, the first Cauchy's inequality may be only strict. But then, after multiplying the reference inequalities, the obtained Inequality (6) must also be strict. Thus, $\left(\frac{n+1}{2}\right)^n >$ $n!$, which was required to be proved.

Example 7. Prove that if $a > 0$, $b > 0$, $c > 0$, then

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geqslant 9. \tag{7}$$

Proof. Let us take the following inequalities as reference ones:

$$\frac{a}{b}+\frac{b}{a} \geqslant 2;\quad \frac{a}{c}+\frac{c}{a} \geqslant 2;\quad \frac{b}{c}+\frac{c}{b} \geqslant 2$$

(these inequalities become equalities in the cases, when, respectively, $a=b$, $a=c$ and $b=c$). Adding them together, we get: $\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b} \geqslant 6$ or $\frac{b+c}{a}+\frac{a+c}{b}+\frac{a+b}{c} \geqslant 6$.

Then, we carry out a number of simple transformations:

$$\left(1+\frac{a+c}{b}\right)+\left(1+\frac{b+c}{a}\right)+\left(1+\frac{a+b}{c}\right) \geqslant 9,$$
$$\frac{a+b+c}{b}+\frac{a+b+c}{a}+\frac{a+b+c}{c} \geqslant 9.$$

Now, taking out of the brackets $a+b+c$, we get:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geqslant 9.$$

The equality sign takes place only if $a=b=c$.

Example 8. Prove that if $n \in N$, $n > 1$, then

$$\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{n^2} < 1. \tag{8}$$

Proof. We have

$$\frac{1}{4}=\frac{1}{2\times 2}<\frac{1}{1\times 2};\quad \frac{1}{9}=\frac{1}{3\times 3}<\frac{1}{2\times 3};$$

$$\frac{1}{16}=\frac{1}{4\times 4}<\frac{1}{3\times 4};\quad \ldots;\ \frac{1}{n^2}=\frac{1}{n\times n}<\frac{1}{(n-1)\,n}.$$

Adding together these $(n-1)$ inequalities, we get:

$$\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{n^2}<\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{(n-1)\,n}$$

$$=\frac{2-1}{1\times 2}+\frac{3-2}{2\times 3}+\frac{4-3}{3\times 4}+\ldots+\frac{n-(n-1)}{(n-1)\,n}$$

$$=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$=1-\frac{1}{n}<1.$$

Thus, $\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots+\frac{1}{n^2}<1.$

3. Proving Inequalities by Contradiction.

Example 9. Prove that if $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, $d\geqslant 0$, then

$$\sqrt{(a+c)(b+d)}\geqslant\sqrt{ab}+\sqrt{cd}. \tag{9}$$

Proof. Suppose that for some values of a, b, c, d Inequality (9) is not true, that is, the inequality $\sqrt{(a+c)(b+d)}<\sqrt{ab}+\sqrt{cd}$ is fulfilled.

Since both sides of this inequality are nonnegative, squaring them, we get:

$$(a+c)(b+d)<ab+cd+2\sqrt{abcd},$$

whence

$$bc+ad<2\sqrt{abcd},$$

and further

$$\frac{bc+ad}{2}<\sqrt{(bc)\times(ad)}.$$

But this contradicts Cauchy's inequality which means that our supposition is not true, and therefore Inequality (9) is true.

Example 10. Prove that if $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, then

$$\frac{a+b+c}{3}\leqslant\sqrt{\frac{a^2+b^2+c^2}{3}}. \tag{10}$$

Proof. Suppose that for some values of a, b, c Inequality (10) is not true, that is, the inequality

$$\frac{a+b+c}{3}>\sqrt{\frac{a^2+b^2+c^2}{3}}$$

is fulfilled. Squaring both members of this inequality, we get:

$$\left(\frac{a+b+c}{3}\right)^2 > \frac{a^2+b^2+c^2}{3},$$

and further

$$(a + b + c)^2 > 3\,(a^2 + b^2 + c^2),$$
$$3\,(a^2 + b^2 + c^2) - (a + b + c)^2 < 0,$$
$$3\,(a^2 + b^2 + c^2) - (a^2 + b^2 + c^2 + 2ab + 2ac + 2bc) < 0,$$
$$2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc < 0,$$
$$(a - b)^2 + (b - c)^2 + (a - c)^2 < 0.$$

The last inequality is not true, since the sum of squares cannot be a negative number. Hence our supposition is false, and therefore Inequality (10) is true.

Remark. Let there be given n nonnegative numbers $a_1, a_2, \ldots, a_n$. We introduce into consideration the following quantities:

$$H_n = \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}$$ — harmonic mean,

$$G_n = \sqrt[n]{a_1 \cdot a_2 \cdot \ldots \cdot a_n}$$ — geometric mean,

$$A_n = \frac{a_1+a_2+\ldots+a_n}{n}$$ — arithmetic mean,

$$Q_n = \sqrt{\frac{a_1^2+a_2^2+\ldots+a_n^2}{n}}$$ — mean square of the numbers $a_1, a_2, \ldots, a_n$.

These quantities are related as follows:

$$H_n \leqslant G_n \leqslant A_n \leqslant Q_n. \qquad (*)$$

Certain particular cases of this relationship were already proved in Items 1-3 of this section. Thus, in Examples 1 and 5 we proved the inequalities $G_2 \leqslant A_2$ and $G_4 \leqslant A_4$; the inequality which is proved in Example 7 implies the relationship $H_3 \leqslant A_3$; finally, the inequality $A_3 \leqslant Q_3$ is proved in Example 10.

4. Proving Inequalities by the Method of Mathematical Induction.

Example 11. Prove that if $n \in N$, $n \geqslant 3$, then

$$2^n > 2n + 1. \qquad (11)$$

Proof. For $n = 3$ Inequality (11) is true: $2^3 > 2 \times 3 + 1$. Let us assume that Inequality (11) is fulfilled for $n = k$ $(k > 3)$, that is, $2^k > 2k + 1$, and let us prove that then Inequality (11) is also fulfilled for $n = k + 1$, that is, prove that $2^{k+1} > 2k + 3$.

Indeed, we have: $2^{k+1} = 2 \times 2^k > 2\,(2k + 1) = 4k + 2 = (2k + 3) + (2k - 1)$. Thus, $2^{k+1} > (2k + 3) + (2k - 1)$.

But $2k - 1 > 0$ for any natural k. Consequently, the more so $2^{k+1} > 2k + 3$.

According to the principle of mathematical induction, we may conclude that Inequality (11) is true for all $n \geqslant 3$.

Example 12. Prove that if $n \in N$, then

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^n - 1} > \frac{n}{2}. \tag{12}$$

Proof. The expression on the left-hand side of Inequality (12) represents the sum of fractions whose denominators are natural numbers from 1 to $2^n - 1$. For $n = 1$ it turns into a true numerical inequality: $1 > \frac{1}{2}$.

Suppose Inequality (12) is fulfilled for $n = k$, that is,

$$S_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^k - 1} > \frac{k}{2}.$$

Let us prove that then Inequality (12) is also true for $n = k + 1$, that is,

$$S_{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^{k+1} - 1} > \frac{k+1}{2}.$$

Indeed, $S_{k+1} = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^k - 1}\right) + \left(\frac{1}{2^k} + \frac{1}{2^k + 1} + \ldots + \frac{1}{2^{k+1} - 1}\right) = S_k + P_k$, where $P_k = \frac{1}{2^k} + \frac{1}{2^k + 1} + \ldots + \frac{1}{2^{k+1} - 1}$.

The expression P_k represents the sum of 2^k fractions each of which is greater than $\frac{1}{2^{k+1}}$. Hence,

$$P_k = \frac{1}{2^k} + \frac{1}{2^k + 1} + \ldots + \frac{1}{2^{k+1} - 1} > \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \ldots + \frac{1}{2^{k+1}}$$
$$= 2^k \times \frac{1}{2^{k+1}} = \frac{1}{2}.$$

Thus, $S_k > \frac{k}{2}$, $P_k > \frac{1}{2}$. But then

$$S_{k+1} = S_k + P_k > \frac{k}{2} + \frac{1}{2} = \frac{k+1}{2}, \text{ i.e. } S_{k+1} > \frac{k+1}{2}.$$

On the basis of the principle of mathematical induction, we conclude that Inequality (12) is true for any $n \in N$.

EXERCISES

In Problems **216** through **268**, prove the given inequalities:

216. $\frac{a^2}{1 + a^4} \leqslant \frac{1}{2}$.

217. If $a \geqslant 0$, $b \geqslant 0$, then $\sqrt{a^5 + b^5} \geqslant a^4 b + ab^4$.

218. If $a>0$, $b>0$, then $\frac{a}{\sqrt{b}}+\frac{b}{\sqrt{a}}\geqslant\sqrt{a}+\sqrt{b}$.

219. If $a+b\geqslant 0$, $a\neq 0$, $b\neq 0$, then $\frac{a}{b^2}+\frac{b}{b^2}\geqslant\frac{1}{a}+\frac{1}{b}$.

220. If $a+b\geqslant 0$, then $ab(a+b)\leqslant a^3+b^3$.

221. $a^2+2b^2+2ab+b+10>0$. 222. $1+2a^4\geqslant a^2+2a^3$.

223. If $a\neq 2$, then $\frac{1}{a^2-4a+4}>\frac{2}{a^3-8}$.

224. If $a\geqslant -1$, then $a^3+1\geqslant a^2+a$. 225. $a^2+b^2+c^2+3\geqslant 2(a+b+c)$.

226. If $a+b\geqslant 0$, then $\frac{a^3+b^3}{2}\geqslant\left(\frac{a+b}{2}\right)^3$.

227. $a^2+b^2\geqslant ab$. 228. $a^4+b^4\geqslant a^3b+ab^3$.

229. $(a+b)^4\geqslant a^4+b^4$, where $ab\geqslant 0$.

230. If $a<b<c$, then $a^2b+b^2c+c^2a<a^2c+b^2a+c^2b$.

231. $a^8-a^5+a^2-a+1>0$.

232. If $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, then $a+b+c\geqslant\sqrt{ab}+\sqrt{bc}+\sqrt{ac}$.

233. If m, n, k are natural numbers, then $mn+mk+nk\leqslant 3mnk$.

234. If $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, then $(a+b)(b+c)(a+c)\geqslant 8abc$.

235. If $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, $a+b+c=1$, then $(1-a)(1-b)(1-c)\geqslant 8abc$.

236. If $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, then $(a+1)(b+1)(c+a)(b+c)\geqslant 16abc$.

237. If $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, $d\geqslant 0$, then $a^4+b^4+c^4+d^4\geqslant 4abcd$.

238. If $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, $d\geqslant 0$, then $\sqrt{(a+b)(c+d)}\leqslant 0.5(a+c)+0.5(b+d)$.

239. If $a_1\geqslant 0$, $a_2\geqslant 0$, $\ldots$, $a_n\geqslant 0$, then $\sqrt{a_1a_2}+\sqrt{a_1a_3}+\ldots+\sqrt{a_1a_n}+\sqrt{a_2a_3}+$

$$\ldots+\sqrt{a_2a_n}+\ldots+\sqrt{a_{n-1}a_n}\leqslant\frac{n-1}{2}(a_1+a_2+\ldots+a_n).$$

240. $\log_2 3+\log_3 2>2$.

241. $\frac{a^2+2}{\sqrt{a^2+1}}\geqslant 2$. 242. $\frac{a^2+a+2}{\sqrt{a^2+a+1}}\geqslant 2$.

243. If $a>0$, $b>0$, $c>0$, then $\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\geqslant a+b+c$.

244. If $a\geqslant 0$, $b\geqslant 0$, $c\geqslant 0$, then $ab(a+b)+bc(b+c)+ac(a+c)\geqslant 6abc$.

245. If $a>0$, $b>0$, then $\frac{a+b}{1+a+b}<\frac{a}{1+a}+\frac{b}{1+b}$.

246. If a_1, a_2, $\ldots$, a_n are nonnegative numbers and $a_1\cdot a_2\cdot\ldots\cdot a_n=1$, then $(1+a_1)(1+a_2)\ldots(1+a_n)\geqslant 2^n$.

247. If $n=2, 3, 4, \ldots$, then $\sqrt{1}+\sqrt{2}+\ldots+\sqrt{n}>n$.

248. If $n=2, 3, 4, \ldots$, then $n!\geqslant n^{n/2}$.

249. If $n=2, 3, 4, \ldots$, then $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{1}{2}$.

250. If $a>0$, $b>0$, then $\frac{2}{\frac{1}{a}+\frac{1}{b}}\leqslant\sqrt{ab}$.

251. If $a \geqslant 0$, $b \geqslant 0$, then $\frac{a+b}{2} \leqslant \sqrt{\frac{a^2+b^2}{2}}$.

252. If $a > 0$, $b > 0$, then $\sqrt{a}+\sqrt{b} > \sqrt{a+b}$.

253. If $a > 0$, $b > 0$, then $\frac{2\sqrt{ab}}{\sqrt{a}+\sqrt{b}} \leqslant \sqrt[4]{ab}$.

254. $\sqrt{a^2+b^2} > \sqrt[3]{a^3+b^3}$.

255. $a^2+b^2+c^2 \geqslant ab+ac+bc$.

256. If $a \geqslant 0$, $b \geqslant 0$, then $(\sqrt{a}+\sqrt{b})^8 \geqslant 16ab\,(a+b)^2$.

257. If $a \geqslant 0$, $b \geqslant 0$, $c \geqslant 0$, then $\frac{a+b+c}{3} \geqslant \sqrt[3]{abc}$.

258. If $abc \neq 0$, $ab+ac+bc \neq 0$, then $\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \leqslant \sqrt[3]{abc}$.

259. $(a+b+c+d)/4 \leqslant \sqrt{(a^2+b^2+c^2+d^2)/4}$.

260. If $a > 0$, $b > 0$, $c > 0$, $d > 0$, then $\frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}} \leqslant \sqrt[4]{abcd}$.

261*. If $x > -1$, $n \geqslant 2$, then $(1+x)^n > 1+nx$.

262. If $n \geqslant 5$, then $2^n > n^2$. 263. If $n \geqslant 10$, then $2^n > n^3$.

264. $|a_1+a_2+\ldots+a_n| \leqslant |a_1|+|a_2|+\ldots+|a_n|$.

265. If $n \geqslant 2$, then $\sqrt{n} < 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}$.

266. If $n \geqslant 2$, then $2\sqrt{n} > 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}}$.

267. If $n \geqslant 2$, then $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n} > \frac{13}{24}$.

268. If $n \geqslant 2$, then $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^n-1} < n$.

SEC. 6. COMPARING NUMERICAL EXPRESSIONS

If two real numbers are given, then in most cases it is clear at once which of them is greater, for instance, $8 > 3$, $\sqrt{6} > \sqrt{5}$. It is not difficult to ascertain that $\sqrt[5]{5} < \sqrt[7]{1000}$. Indeed, $\sqrt[5]{5} < 2$, and $\sqrt[7]{1000} > 2$, hence, $\sqrt[5]{5} < \sqrt[7]{1000}$.

Let now $a = \sqrt[3]{3}$, $b = \sqrt{2}$. Both numbers belong to the interval $(1, 2)$, but it is not clear, for the time being, which of them is greater. To determine the inequality sign between these numbers, let us reason in the following way. Suppose that $a > b$, that is, that $\sqrt[3]{3} > \sqrt{2}$. Raising both sides of the last inequality to the sixth power, we get $(\sqrt[3]{3})^6 > (\sqrt{2})^6$, i.e. $9 > 8$.

* In Problems 261 through 268, it is assumed that $n \in N$.

Thus, $a > b \Leftrightarrow (\sqrt[3]{3})^6 > (\sqrt{2})^6 \Leftrightarrow 9 > 8$.

Since $9 > 8$ is a true inequality, the equivalent inequality $a > b$ is also true.

If we assumed that $a < b$, then we would get: $a < b \Leftrightarrow (\sqrt[3]{3})^6 < (\sqrt{2})^6 \Leftrightarrow 9 < 8$. Since $9 < 8$ is a false inequality, $a < b$ is also not true, and since $a \neq b$, it remains only one possibility: $a > b$.

Example 1. Compare the numbers a and b if: (1) $a = \sqrt{26} + \sqrt{6}$, $b = \sqrt{13} + \sqrt{17}$; (2) $a = \log_{0.5} 3$, $b = \log_3 1.1$; (3) $a = \log_2 3$, $b = \log_3 2$; (4) $a = \sqrt{5} + \sqrt{30} + \sqrt{50}$, $b = \sqrt{10} + \sqrt{20} + \sqrt{60}$.

Solution. (1) Let us assume that $a > b$. Then, using the properties of numerical inequalities, we get in succession:

$$(\sqrt{26}+\sqrt{6})^2 > (\sqrt{13}+\sqrt{17})^2,\ 32+2\sqrt{156} > 30+2\sqrt{221},$$

$$1+\sqrt{156} > \sqrt{221},\ (1+\sqrt{156})^2 > 221,\quad \sqrt{156} > 32.$$

Thus, $a > b \Leftrightarrow \sqrt{156} > 32$. But $\sqrt{156} < 32$, hence, the original numbers are connected with the same inequality sign, that is, $a < b$.

(2) We have: $a = \log_{\frac{1}{2}} 3 < \log_{\frac{1}{2}} 1 = 0$, $b = \log_3 1.1 > \log_3 1 = 0$. Hence, $a < 0$, $b > 0$, i.e. $a < b$.

(3) We have: $a = \log_2 3 > \log_2 2 = 1$, $b = \log_3 2 < \log_3 3 = 1$. Thus, $a > 1$, $b < 1$, hence, $a > b$.

(4) We first bound each term of the sum a by the nearest integers. We have: $2 < \sqrt{5} < 3$, $5 < \sqrt{30} < 6$, $7 < \sqrt{50} < 8$, that is, $14 < \sqrt{5} + \sqrt{30} + \sqrt{50} + 17$. Thus, $14 < a < 17$.

Now, we bound each term of the sum b also by the nearest integers. We have: $3 < \sqrt{10} < 4$, $4 < \sqrt{20} < 5$, $7 < \sqrt{60} < 8$, that is, $14 < \sqrt{10} + \sqrt{20} + \sqrt{60} < 17$. Thus, $14 < b < 17$.

We have obtained the same bounds for the numbers a and b and so we cannot compare them.

Let us increase the precision of each bound of the sums a and b, taking each bound to an accuracy of 0.1. We have

$$\begin{array}{rcl} 2.2 < \sqrt{5} < 2.3 & & 3.1 < \sqrt{10} < 3.2 \\ 5.4 < \sqrt{30} < 5.5 & \text{and} & 4.4 < \sqrt{20} < 4.5 \\ 7 < \sqrt{50} < 7.1 & & 7.7 < \sqrt{60} < 7.8 \\ \hline 14.6 < a \quad < 14.9 & & 15.2 < b \quad < 15.5 \end{array}$$

Thus, $a \in (14.6,\ 14.9)$ and $b \in (15.2,\ 15.5)$. Hence, $a < b$.

Example 2. Arrange the following numbers in the increasing order:

$$a = \log_2 3, \quad b = \log_6 9, \quad c = \log_5 17.$$

Solution. We compare the numbers a and b. This can be done by two methods.

First Method. We have: $\log_2 2 < \log_2 3 < \log_2 4$, that is, $1 < a < 2$; $\log_6 6 < \log_6 9 < \log_6 36$, that is, $1 < b < 2$.

The numbers a and b belong to the interval $(1, 2)$. Let us compare each of them with the middle of the interval, that is, with the number $\frac{3}{2}$.

Suppose that $\log_2 3 > \frac{3}{2}$, then we shall have in succession: $3 > 2^{\frac{3}{2}} \Leftrightarrow 3^2 > 2^3 \Leftrightarrow 9 > 8$. And since $a > \frac{3}{2} \Leftrightarrow 9 > 8$, then $a > \frac{3}{2}$ is a true inequality. Suppose that $b > \frac{3}{2}$, then $\log_6 9 > \frac{3}{2} \Leftrightarrow 9 > 6^{\frac{3}{2}} \Leftrightarrow 9^2 > 6^3 \Leftrightarrow 81 > 216$. The last inequality is not true, and since $b > \frac{3}{2} \Leftrightarrow 81 > 216$, the inequality $b > \frac{3}{2}$ is not true either. Hence, $b < \frac{3}{2}$.

Thus, $a > \frac{3}{2}$, $b < \frac{3}{2}$, hence, $a > b$.

Second Method. Consider the difference $a - b$. We have:

$$a - b = \log_2 3 - \log_6 9 = \log_2 3 - \frac{\log_2 9}{\log_2 6}$$

$$= \frac{\log_2 3 \ \log_2 6 - 2\log_2 3}{\log_2 6} = \frac{\log_2 3\,(\log_2 6 - 2)}{\log_2 6} > 0.$$

Hence, $a > b$.

Let us now compare the numbers a and c. It was established above that $\frac{3}{2} < a < 2$. The number c is also enclosed in these bounds. Indeed, $\log_5 17 < \log_5 25 = 2$. On the other hand,

$$\frac{3}{2} = \log_5 5^{\frac{3}{2}} = \log_5 \sqrt{125} < \log_5 17.$$

Let us compare the numbers a and c with the middle of the interval $\left(\frac{3}{2}, 2\right)$, that is, with the number $\frac{7}{4}$.

Suppose that $a > \frac{7}{4}$. Then, using the properties of inequalities, we get in succession:

$$\log_2 3 > \frac{7}{4} \Leftrightarrow 3 > 2^{\frac{7}{4}} \Leftrightarrow 3^4 > 2^7 \Leftrightarrow 81 > 128.$$

Indeed, $81 < 128$, and hence, $a < \frac{7}{4}$.

Suppose that $c > \frac{7}{4}$. Then

$$\log_5 17 > \frac{7}{4} \Leftrightarrow 17 > 5^{\frac{7}{4}} \Leftrightarrow 17^4 > 5^7.$$

The last inequality is true, hence our supposition is true.

Thus, $a < \frac{7}{4}$, $c > \frac{7}{4}$, and therefore $a < c$. Hence $b < a < c$.

EXERCISES

In Problems 269 through 283, compare the numbers a and b:

269. $a = \sqrt[5]{5}$, $b = \sqrt[6]{6}$. **270.** $a = \sqrt{47}$, $b = \sqrt{26} + \sqrt{6}$.

271. $a = 1 + \frac{1}{\sqrt{2}}$, $b = 2(\sqrt{2} - 1)$. **272.** $a = 6$, $b = \frac{3\sqrt{7} + 5\sqrt{2}}{\sqrt{5}}$.

273. $a = \sqrt[4]{9 - \sqrt{15}}$, $b = \sqrt{\frac{\sqrt{30} - \sqrt{2}}{2}}$.

274. $a = \sqrt[4]{79 + \sqrt[3]{26}}$, $b = \sqrt[4]{84 - \sqrt[3]{28}}$.

275. $a = \sqrt{3} + \sqrt{23} + \sqrt{53}$, $b = \sqrt{13} + \sqrt{33} + \sqrt{43}$.

276. (a) $a = \log_4 2$, $b = \log_{0.0625} 0.25$; (b) $a = \log_4 5$, $b = \log_{\frac{1}{16}} \frac{1}{25}$.

277. (a) $a = \log_4 26$, $b = \log_6 17$; (b) $a = \log_{\frac{1}{2}} \sqrt{3}$, $b = \log_{\frac{1}{3}} \sqrt{2}$.

278. (a) $a = \log_2 3$, $b = \log_5 8$; (b) $a = \log_3 16$, $b = \log_{16} 729$.

279. $a = \log_5 14$, $b = \log_7 18$. **280.** $a = \log_{20} 80$, $b = \log_{80} 640$.

281. $a = \frac{\log 5 + \log \sqrt{7}}{2}$, $b = \log \frac{5 + \sqrt{7}}{2}$.

282. $a = 3(\log 7 - \log 5)$, $b = 2\left(\frac{1}{2}\log 9 - \frac{1}{3}\log 8\right)$.

283. $a = \frac{1}{\log_2 \pi} + \frac{1}{\log_{4.5} \pi}$, $b = 2$.

284. Arrange in the increasing order the numbers a, b, c, d if $a = \log_5 7$, $b = \log_8 3$, $c = \sqrt{2}$, $d = \log_{\frac{1}{4}} 5$.

Chapter 2

SOLVING EQUATIONS AND INEQUALITIES

SEC. 7. EQUIVALENT EQUATIONS

Two equations are said to be *equivalent* if the sets of their roots coincide, in particular, if both equations have no roots.

For instance, the equations $\log x = 0$ and $\sqrt{x} = 1$ are equivalent (each of them has the only root $x = 1$); the equations $2^{x(x-1)} = 1$ and $\sqrt{x} = x$ are also equivalent (each of them has two roots: 0 and 1).

If each root of the equation $f(x) = g(x)$ is at the same time a root of the equation $f_1(x) = g_1(x)$ obtained by some transformations from the equation $f(x) = g(x)$, then the equation $f_1(x) = g_1(x)$ is called the *consequence* of the equation $f(x) = g(x)$.

Thus, the equation $(x - 1)(x - 2) = 0$ is a consequence of the equation $x - 1 = 0$ (whereas the equation $x - 1 = 0$ is not a consequence of the equation $(x - 1)(x - 2) = 0$).

If each of two equations is a consequence of the other, then such equations are equivalent.

Several equations in one variable are called a *collection* of equations if a problem is posed to find all such values of the variable each of which satisfies at least one of the given equations. Equations forming a collection are written in the following manner:

$$\left[\begin{array}{l} 2x + 1 = 3x + 5 \\ 4x - 3 = x^2 \end{array}\right.$$

(however, more frequently, the equations forming a collection are written in line: $2x + 1 = 3x + 5$; $4x - 3 = x^2$ and separated by a semicolon).

The *solution of a collection* of equations is defined as the *union* of the sets of the roots of the equations forming the collection.

If, as a result of transformations, the equation $f(x) = g(x)$ is reduced to the equation $f_1(x) = g_1(x)$ (or to a collection of equations) some roots of which are not roots of the equation $f(x) = g(x)$, then the roots of the equation $f_1(x) = g_1(x)$ are said to be *extraneous* roots of the equation $f(x) = g(x)$.

For instance, squaring both sides of the equation $\sqrt{x} = -x$, we get the equation $x = x^2$ having two roots: 0 and 1. The value

$x = 0$ satisfies the equation $\sqrt{x} = -x$, whereas $x = 1$ does not satisfy this equation, that is, this value is an extraneous root of the equation $\sqrt{x} = -x$.

The equation $(x - 1)^2 = x - 1$ has two roots: 1 and 2. Dividing both sides of this equation by $x - 1$, we get the equation $x - 1 = 1$ which has only one root: $x = 2$. In such cases we say that in the process of transforming the original equation there happened a *loss of roots* (in our example $x = 1$ is a "lost root").

When solving equations, we usually perform various transformations which reduce a given equation to a simpler one (or to a collection of equations). Therefore it is important to know which of the transformations reduces the given equation to an equivalent or a consequent equation, and which results in a loss of roots.

Theorem 1. *If the function $\varphi(x)$, defined for all x's from the domain of definition of the equation $f(x) = g(x)$, is added to both sides of this equation, then we get the equation $f(x) + \varphi(x) = g(x) + \varphi(x)$ which is equivalent to the given one.*

For instance, the equation $3x^2 + 2x - 5 = 7x - 1$ is equivalent to $3x^2 + 2x - 5 + (-7x + 1) = 7x - 1 + (-7x + 1)$ since the function $\varphi(x) = -7x + 1$ is defined for all values of x from the domain of definition of the equation $3x^2 + 2x - 5 = 7x - 1$.

But the equation $x^2 = 1$ is not equivalent to the equation $x^2 + \sqrt{x} = 1 + \sqrt{x}$. Here, the equivalence is violated since the function $\varphi(x) = \sqrt{x}$ is defined not for all x's from the domain of definition of the equation $x^2 = 1$, but only for $x \geqslant 0$. By adding the expression $\varphi(x) = \sqrt{x}$ to both sides of the equation $x^2 = 1$, we reduced the domain of definition of the equation which might lead to a loss of solutions. In this case $x = -1$ is a root of the equation $x^2 = 1$, but is not a root of the equation $x^2 + \sqrt{x} = 1 + \sqrt{x}$.

It should be clearly understood that Theorem 1 deals only with one transformation, namely, with adding the same function to both sides of an equation. The subsequent collection of like terms (if it is possible) is a new transformation of the equation. Collecting like terms can lead to an equation which is not equivalent to the original one. For example, adding the function $\varphi(x) = -\log x$ to both sides of the equation $x^2 + 2x + \log x = \log x - 1$, we get the equation $x^2 + 2x + \log x - \log x = \log x - 1 - \log x$ which is equivalent to the original one since the function $\varphi(x) = -\log x$ is defined for all x's from the domain of definition of the original equation. But collecting like terms in the newly obtained equation, we get the equation $x^2 + 2x = -1$ which is not equivalent to the original one. The annihilation of $\log x$ in both sides of the given equation has led to an *extension of the domain of definition* of the equation which may result in appearance of extraneous roots. This has just happened in

our case: the value $x = -1$ is a root of the equation $x^2 + 2x = -1$, but is not a root of the equation $x^2 + 2x + \log x = \log x - 1$.

Corollary. *The equations* $f(x) + \varphi(x) = g(x)$ and $f(x) = g(x) - \varphi(x)$ *are equivalent.*

Theorem 2. *If both sides of the equation* $f(x) = g(x)$ *are multiplied or divided by the same function* $\varphi(x)$ *which is defined for all values of* x *from the domain of definition of the equation and vanishes nowhere in this domain, then the following equation is obtained*

$$f(x)\,\varphi(x) = g(x)\,\varphi(x) \left(\text{or } \frac{f(x)}{\varphi(x)} = \frac{g(x)}{\varphi(x)}\right),$$

which is equivalent to the given equation.

Thus, dividing both sides of the equation $x - 4 = x(\sqrt{x} + 2)$ by $\varphi(x) = \sqrt{x} + 2$, we get the equation $\sqrt{x} - 2 = x$ which is equivalent to the given one since the function $\varphi(x) = \sqrt{x} + 2$ is defined everywhere in the domain of definition of the given equation ($x \geqslant 0$) and vanishes nowhere in this domain.

Multiplying both sides of the equation $x - 2 = 0$ by $\varphi(x) = x + 3$, we get the equation $(x - 2)(x + 3) = 0$ which is not equivalent to the given one since for $x = -3$, from the domain of definition of the original equation, the function $\varphi(x) = x + 3$ vanishes although it is defined for all x's from the domain of definition of the equation $x - 2 = 0$. As is easily seen, in this case the multiplication of both sides of the equation by $\varphi(x) = x + 3$ has led to the appearance of the extraneous root $x = -3$.

Similarly, dividing both sides of the equation $x - 4 = x(\sqrt{x} - 2)$ by $\varphi(x) = \sqrt{x} - 2$, we get the equation $\sqrt{x} + 2 = x$ which is not equivalent to the given equation since for $x = 4$ the function $\varphi(x) = \sqrt{x} - 2$ vanishes although it is defined for all x's from the domain of definition of the original equation.

We call the reader's attention to the fact that Theorem 2 deals only with one transformation, namely, with multiplication (or division) of both sides of an equation by the same function. The subsequent reduction of the fraction (if it is possible) is already a new transformation of the equation. Thus, multiplying both sides of the equation $\frac{x+1}{2x} + x = 3$ by the function $\varphi(x) = 2x$, we complete the first transformation which leads to the equation $\frac{2x(x+1)}{2x} + 2x^2 = 6x$. The subsequent reduction of the fraction $\frac{2x(x+1)}{2x}$ by $2x$ is regarded as a new transformation: it leads to the equation $x + 1 + 2x^2 = 6x$. This reduction can also lead to an equation not equivalent to the original one.

Thus, multiplying both sides of the equation $\frac{x^2 - 5x + 6}{x - 2} = 0$ by $\varphi(x) = x - 2$, we get the equation $\frac{(x^2 - 5x + 6)(x - 2)}{x - 2} = 0$ which is equivalent to the given one since the function $\varphi(x) = x - 2$ is defined for all values of x from the domain of definition of the given equation ($x \neq 2$) and vanishes nowhere in this domain of definition. But reducing the left-hand side of the obtained equation by $x - 2$, we get the equation $x^2 - 5x + 6 = 0$ which is not equivalent to the given one: the value $x = 2$ is a solution of the last equation but does not satisfy the original equation, being its extraneous root. The thing is that here the domain of definition is extended due to the reduction of the fraction, which, as we have already noted, can lead to extraneous roots.

Corollary. *If both sides of an equation are multiplied (or divided) by the same number different from zero, then an equation equivalent to the given equation is obtained.*

For instance, multiplying both sides of the equation $\frac{x + 1}{2} = \frac{x + 3}{3}$ by 6, we get the equation $3x + 3 = 2x + 6$ which is equivalent to the given one.

Theorem 3. *If both sides of the equation* $f(x) = g(x)$, *where* $f(x) \times g(x) \geqslant 0$ *for all values of* x *from the domain of definition of the equation, are raised to the same natural power* n, *then the equation* $(f(x))^n = (g(x))^n$ *is obtained which is equivalent to the given equation.*

For instance, squaring both sides of the equation $2x - 1 = \sqrt{x - 1}$, we get the equation $(2x - 1)^2 = (\sqrt{x - 1})^2$ which is equivalent to the given one since for all x's from the domain of definition of the given equation ($x \geqslant 1$) both sides of the equation are nonnegative.

An opposite example: squaring both sides of the equation $x - 6 = \sqrt{x}$, we get the equation $(x - 6)^2 = (\sqrt{x})^2$ which cannot be said to be equivalent to the given one since for some values of x from the domain of definition of the given equation ($x \geqslant 0$) the left-hand side of the equation takes on negative values (for $x = 2$, we have: $x - 6 = -4 < 0$) while the right-hand side is always nonnegative. Indeed, the equation $(x - 6)^2 = (\sqrt{x})^2$ is transformed to $x^2 - 13x + 36 = 0$, whence $x_1 = 9$, $x_2 = 4$. But $x = 4$ is an extraneous root of the original equation.

Note that Theorem 3 deals only with one transformation, namely, with raising both sides of the equation to the same natural power. The subsequent rationalization (if it is possible) is already a new transformation of the equation. Getting rid of radicals can result in extending the domain of definition of the equation, and therefore can lead to an equation not equivalent to the original one.

Remarks: 1. Theorem 3 holds true only for equations defined in the field of real numbers.

2. If n is an odd number, then in the statement of Theorem 3 we may omit the condition: $f(x)\, g(x) \geqslant 0$ for all x's from the domain of definition of the equation.

When solving equations, we also have to use transformations not stipulated by Theorems 1, 2, and 3, that is, transformations which can lead to the appearance of extraneous roots or even to a loss of roots. This can be caused by the transformations carried out by formulas changing the domains of definition of equations. Here are examples of such formulas:

$$\sqrt{ab}=\sqrt{a}\sqrt{b}, \quad \sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}, \quad (\sqrt{a})^2=a,$$

$$\log_a(xy)=\log_a x+\log_a y, \quad a^{\log_a b}=b,$$

$$\tan x \cot x = 1, \quad \sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},$$

$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y} \text{ and so on.}$$

In all cases when the carried-out transformations lead to the consequence of the given equation, but when we are not sure that both equations are equivalent, the found solutions must be checked. Such a check is an integral part of solving an equation. The process of solving cannot be regarded as completed if no check is carried out.

What technique can be used to check the found solutions? Two basic methods may be indicated: (1) substituting each of the found solutions into the original equation; (2) proving the equivalence of the completed transformations of an equation throughout all steps of solving.

Example 1. Solve the equation $\sqrt{2x+5}=8-\sqrt{x-1}$.

Solution. Squaring both sides of the equation, we get $2x+5=(8-\sqrt{x-1})^2$, and further $16\sqrt{x-1}=58-x$.

The obtained equation is squared once again: $256(x-1)=(58-x)^2$, and further

$$x^2-372x+3620=0,$$

whence $x_1=10$, $x_2=362$.

Analysing the above transformations, we may assert only that each new equation was a consequence of its predecessor. (We are not sure that the equations obtained in the process of solving are equivalent.) But this means that extraneous roots might appear in the process of solving, and therefore the found roots must be checked.

Check. In the present case, it is not difficult to check the found roots by substituting them into the original equation. Let us check $x_1 = 10$. We have: $\sqrt{2x_1 + 5} = \sqrt{2 \times 10 + 5} = 5$ and $8 - \sqrt{x_1 - 1} = 8 - \sqrt{10 - 1} = 5$. Thus, for $x = 10$ both sides of the original equation take on equal numerical values, hence, $x = 10$ is a root of the given equation.

We now check $x_2 = 362$. We have: $\sqrt{2x_2 + 5} = \sqrt{2\times 362 + 5} = 27$, and $8 - \sqrt{x_2 - 1} = 8 - \sqrt{362 - 1} = -11$. For $x = 362$ the left-hand and right-hand sides of the original equation take on different numerical values, hence, $x = 362$ is an extraneous root.

Thus, our equation has only one root: $x = 10$.

Example 2. Solve the equation $\sqrt{3x + 1} = 3 + \sqrt{x - 1}$.

Solution. Squaring both sides of the equation, we get:

$$3x + 1 = (3 + \sqrt{x - 1})^2,$$

and further $6\sqrt{x - 1} = 2x - 7$.

Squaring once again, we get: $36(x - 1) = (2x - 7)^2$, and further $4x^2 - 64x + 85 = 0$, whence we find: $x_1 = \frac{16+3\sqrt{19}}{2}$, $x_2 = \frac{16-3\sqrt{19}}{2}$.

Check. It is clear that the check of found roots by substituting them into the original equation involves considerable computational difficulties. Therefore let us choose another method.

The domain of definition of the given equation is $x \geqslant 1$. In this domain the first squaring is an equivalent transformation of the equation. The second squaring is applied to the equation $6\sqrt{x - 1} = 2x - 7$. This equation can only be satisfied by such values of x which satisfy the inequality $2x - 7 \geqslant 0$, that is, $x \geqslant 3.5$. It is easily determined that the inequality $\frac{16 + 3\sqrt{19}}{2} \geqslant 3.5$ is true, and the inequality $\frac{16 - 3\sqrt{19}}{2} \geqslant 3.5$ is false. Hence $x_2 = \frac{16 - 3\sqrt{19}}{2}$ is an extraneous root, and $x_1 = \frac{16 + 3\sqrt{19}}{2}$ is the only root of the given equation.

Example 3. Solve the equation

$$\log(x^2 - 7x + 3) - \log(2x + 1) = \log(x^2 + 7x - 3) - \log(2x - 1).$$

Solution. We transform the given equation to the form:

$$\log\frac{x^2-7x+3}{2x+1} = \log\frac{x^2+7x-3}{2x-1},$$

and further $\frac{x^2-7x+3}{2x+1}=\frac{x^2+7x-3}{2x-1}$, whence we find: $x_1=0$, $x_2=\frac{2}{5}$.

Check. Since each equation obtained at one or another step of solution is only the consequence of its predecessor, no loss of roots might happen, while extraneous roots might appear only due to extension of the domain of definition of the original equation. Therefore, in this case the check may be realized using the domain of definition of the original equation which is specified by the following system of inequalities:

$$\begin{cases} x^2-7x+3>0 \\ 2x+1>0 \\ x^2+7x-3>0 \\ 2x-1>0. \end{cases}$$

Neither $x_1 = 0$ nor $x_2 = \frac{2}{5}$ satisfy the last inequality of the above system, and, hence, they are extraneous roots. Thus, the equation has no roots.

EXERCISES

In Problems 285 through 294, prove that the indicated equations have no roots:

285. $\sqrt{x-1}+\sqrt{2-x}=\sqrt{x-5}$.

286. $\sqrt[4]{x^2-144}=\sqrt{x-8}+\sqrt{8-x}$.

287. $\log_2(x^2-1)+\log_3(x^3-1)+\log_4(1-x^4)=\sqrt{x}$.

288. $2^{\log_{2x}(x+2)}+3^{\log_2(x+3)}=\sqrt{-1-x}$.

289. $\sqrt{x-1}+\sqrt{2-x}=x-5$.

290. $x^2+\frac{1}{x^2-16}=16+\frac{1}{x^2-16}$.

291. $\log(10-x^2)=\sqrt{x}+\sqrt{x+2}$. **292.** $2^{\log_2(x-3)}=2x-5$.

293. $\sqrt{x^2+1}+\sqrt{x^4+1}=1$. **294.** $x^4+x^2+1=\log_{\frac{1}{3}} 2$.

In Problems 295 through 304, find out whether the given pairs of equations are equivalent:

295. $x^2+1=\sqrt{x}$ and $x^2+1+\sqrt{1-x}=\sqrt{x}+\sqrt{1-x}$.

296. $x^2-1=\sqrt{x}$ and $x^2-1+\sqrt{1-x}=\sqrt{x}+\sqrt{1-x}$.

297. $x^3+x=0$ and $\frac{x^3+x}{x}=0$. **298.** $x^2+1=0$ and $\frac{x^2+1}{x}=0$.

299. $\dfrac{2x^2+2x+3}{x+3}=\dfrac{3x^2+2x-1}{x+3}$ and $2x^2+2x+3=3x^2+2x-1$.

300. $\dfrac{2x^2+2x+3}{x+2}=\dfrac{3x^2+2x-1}{x+2}$ and $2x^2+2x+3=3x^2+2x-1$.

301. $\sqrt{x}+2=\sqrt{2x}+1$ and $(\sqrt{x}+2)^2=(\sqrt{2x}+1)^2$.

302. $(\sqrt{x}-2)^2=(\sqrt{2x}+1)^2$ and $x-4\sqrt{x}+4=2x+2\sqrt{2x}+1$.

303. $2\sqrt{x}-7x^2=2\left(\dfrac{x}{2}+\sqrt{x}\right)$ and $2\sqrt{x}-7x^2=2x+2\sqrt{x}$.

304. $2\sqrt{x}-7x^2=2x+2\sqrt{x}$ and $-7x^2=2x$.

In Problems 305 through 313, find out whether the given equations and collections of equations are equivalent (explain your answer):

305. $(x-4)(x+3)=0$ and $x-4=0$; $x+3=0$.

306. $(x-4)\left(x+\dfrac{1}{x+3}\right)=0$ and $x-4=0$; $x+\dfrac{1}{x+3}=0$.

307. $(x-4)\left(x+\dfrac{1}{x-4}\right)=0$ and $x-4=0$; $x+\dfrac{1}{x-4}=0$.

308. $\sqrt{x-2}\sqrt{x+3}=0$ and $\sqrt{x-2}=0$; $\sqrt{x+3}=0$.

309. $\sqrt{2-x}\sqrt{x+3}=0$ and $\sqrt{2-x}=0$; $\sqrt{x+3}=0$.

310. $(x-3)\log(2-x)=0$ and $x-3=0$; $\log(2-x)=0$.

311. $(2-x)\log(x-3)=0$ and $2-x=0$; $\log(x-3)=0$.

312. $\dfrac{(x^2-2x-3)(x+1)}{x-3}=0$ and $x^2-2x-3=0$; $x+1=0$.

313. $\dfrac{x^2-5x+6}{x^2-6x+8}\left(2^{\frac{x+4}{x^2-9}}-1\right)=0$ and $\left[\begin{array}{l} x^2-5x+6=0 \\ 2^{\frac{x+4}{x^2-9}}-1=0. \end{array}\right.$

In Problems 314 through 330, solve the indicated equations and check the results obtained. If there are extraneous roots, find out the cause of their appearance.

314. $\dfrac{2}{2-x}+\dfrac{1}{2}=\dfrac{4}{2x-x^2}$. 315. $\dfrac{x+2}{x+1}+\dfrac{2-x}{1-x}+\dfrac{4}{x-1}=0$.

316. $\dfrac{x}{2x-1}+\dfrac{25}{4x^2-1}=\dfrac{1}{27}-\dfrac{13}{1-2x}$.

317. $\dfrac{6}{x^2-1}-\dfrac{2}{x-1}=2-\dfrac{x+4}{x-1}$. 318. $1+\sqrt{2x+7}=x-3$.

319. $\dfrac{x-2}{\sqrt{2x-7}}=\sqrt{x-4}$. 320. $\sqrt{22-x}-\sqrt{10-x}=2$.

321. $\sqrt{x+3}+\sqrt{3x-2}=7$. 322. $\sqrt{3x-2}=2\sqrt{x+2}-2$.

323. $\sqrt{2x+1}+\sqrt{x-3}=2\sqrt{x}$. 324. $\log(54-x^3)=3\log x$.

325. $\log(x-2)+\log(x-3)=1-\log 5$.

326. $\log \sqrt{5x-4}+\log \sqrt{x+1}=2+\log 0.18$.

327. $\dfrac{\log(3x-5)}{\log(3x^2+25)}=\dfrac{1}{2}$. **328.** $\dfrac{\log(2x-5)}{\log(x^2-8)}=0.5$.

329. $\log_x(2x^2-7x+12)=2$. **330.** $\log_x(2x^2-4x+3)=2$.

SEC. 8. RATIONAL EQUATIONS

This section deals with equations of the form $P(x) = 0$, $\frac{P(x)}{Q(x)} = 0$, where $P(x)$ and $Q(x)$ are polynomials, and also of the form $f(x) = g(x)$, where $f(x)$ and $g(x)$ are rational expressions.

Let us recall some statements from algebra.

1. *Any polynomial of degree n defined in the field of complex numbers has n complex roots.*

2. *If $x = a$ is a root of the polynomial $P(x)$, then $P(x)$ is divisible by the binomial $x - a$.*

3. *Let all the coefficients of the polynomial $P(x)$ be integers, the leading coefficient being equal to 1. If such polynomial has a rational number as its root, then this number is an integer.*

4. *Let all the coefficients of the polynomial $P(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$ be integers. If the integer b is a root of the polynomial, then b is the divisor of the constant term a_n* (the necessary condition for an integral root to exist).

Note that when solving equations with integer exponents and rational coefficients, only equivalent transformations are used, therefore the found roots should not be checked; there is no need to mention about it in each concrete case. When solving fractional rational equations, both sides of an equation are multiplied by the same expression (getting rid of denominators) which may lead to extraneous roots. Therefore, when solving fractional rational equations, a check is necessary.

When solving equations with rational coefficients, the following basic methods are used: (1) factorization; (2) introduction of new (auxiliary) variables.

The factorization method consists in the following: let

$$f(x) = f_1(x) \cdot f_2(x) \cdot \ldots \cdot f_n(x),$$

then any solution of the equation

$$f(x) = 0 \tag{1}$$

is a solution of the collection of equations

$$f_1(x) = 0; \quad f_2(x) = 0; \ldots; f_n(x) = 0. \tag{2}$$

The converse is not true: not any solution of the collection of equations (2) is a solution of Equation (1).

For instance, the equation

$$\frac{x^2-3x+2}{x}\left(\frac{x+2}{x^2-1}+2\right)=0 \tag{3}$$

is reduced to the collection of equations:

$$\frac{x^2-3x+2}{x}=0;\ \frac{x+2}{x^2-1}+2=0. \tag{4}$$

The solutions of Collection (4) are: $x_1 = 1$, $x_2 = 2$, $x_3 = 0$, $x_4 = -\frac{1}{2}$.

But for $x = 1$ the function $\frac{x+2}{x^2-1}$ is not defined, and for $x = 0$ the function $\frac{x^2-3x+2}{x}$ is not defined.

Thus, the values $x = 1$, $x = 0$ are not the roots of Equation (3).

In general, when solving Equation (1) by the factorization method, we should bear in mind that Equation (1) is satisfied by those and only those of the found roots from Collection (2), which belong to the domain of definition of Equation (1).

Example 1. Solve the equation $x^3 + 2x^2 + 3x + 6 = 0$.

Solution. Let us factorize the left-hand side of the equation. We have: $x^2(x + 2) + 3(x + 2) = 0$, and further: $(x + 2)(x^2 + 3) = 0$.

The last equation is equivalent to the collection of equations

$$x + 2 = 0;\quad x^2 + 3 = 0.$$

Solving this collection, we get: $x_1 = -2$, $x_{2,3} = \pm i\sqrt{3}$. These are just roots of the given equation.

Example 2. Solve the equation $x^4 + x^3 + 3x^2 + 2x + 2 = 0$.

Solution. The attempts to group some terms in the left-hand side of the equation, as it was done in Example 1, turn out to be unsuccessful. Therefore let us try to represent some term of the equation in the form of the sum of several terms so that the grouping enabling us to get a "successful" factorization becomes feasible. Let us set $3x^2 = x^2 + 2x^2$.

Then, we get: $(x^4 + x^3 + x^2) + (2x^2 + 2x + 2) = 0$, and further: $x^2(x^2 + x + 1) + 2(x^2 + x + 1) = 0$, $(x^2 + x + 1)(x^2 + 2) = 0$.

It remains to solve the collection of equations: $x^2 + x + 1 = 0$; $x^2 + 2 = 0$.

From this collection we find: $x_{1,2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$, $x_{3,4} = \pm i\sqrt{2}$.

Example 3. Solve the equation $x^3 + 4x^2 - 24 = 0$.

Solution. Let us try to find the integral root of the equation. To this end, we write out the divisors of the constant term:

$$\alpha = \pm 1;\ \pm 2;\ \pm 3;\ \pm 4;\ \pm 6;\ \pm 12;\ \pm 24.$$

We now begin our trials. Substituting the value $\alpha = 1$ for x into the equation, we get: $1^3 + 4 \times 1^2 - 24 \neq 0$. Thus, $x = 1$ is not a root of the equation. We continue our trial for $\alpha = -1 \div (-1)^3 + 4 \times (-1)^2 - 24 \neq 0$, and then for $\alpha = 2 \div 2^3 + 4 \times 2^2 - 24 = 0$. Thus, $x_1 = 2$ is a root of the equation.

Since the given equation is of the third degree, it has two more roots. Let us use that the polynomial $x^3 + 4x^2 - 24$ is divisible by $x - 2$ (without a remainder), the quotient being $x^2 + 6x + 12$.

Thus, $x^3 + 4x^2 - 24 = (x - 2)(x^2 + 6x + 12)$, and, hence, the original equation takes the form:

$$(x - 2)(x^2 + 6x + 12) = 0.$$

This equation is equivalent to the collection of equations (one of which is already solved): $x - 2 = 0$; $x^2 + 6x + 12 = 0$. We find from the second equation of the collection: $x_{2,3} = -3 \pm i\sqrt{3}$.

Thus, the given equation has the following roots: $x_1 = 2$, $x_2 = -3 + i\sqrt{3}$, $x_3 = -3 - i\sqrt{3}$.

Remark. The equation $x^3 + 4x^2 - 24 = 0$ can be solved by the factorization method. Representing $4x^2$ as the sum $-2x^2 + 6x^2$, we get in succession:

$$x^3 - 2x^2 + 6x^2 - 24 = 0,$$

$$x^2(x - 2) + 6(x - 2)(x + 2) = 0, \quad \text{and so on.}$$

Example 4. Solve the equation $x^6 - 9x^3 + 8 = 0$.

Solution. Let us apply the method of introducing a new variable. We set $y = x^3$. Then the given equation takes the form: $y^2 - 9y + 8 = 0$, whence we find: $y_1 = 1$, $y_2 = 8$. Now the problem is reduced to solving the collection of equations: $x^3 = 1$; $x^3 = 8$.

Let us solve the first equation. We have: $x^3 - 1 = 0$, and further:

$(x - 1)(x^2 + x + 1) = 0$, whence $x_1 = 1$, $x_{2,3} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$.

Similarly, from the second equation of the collection we find:

$x_4 = 2$, $x_{5,6} = -1 \pm i\sqrt{3}$.

Example 5. Solve the equation

$$(x^2 + x + 4)^2 + 8x(x^2 + x + 4) + 15x^2 = 0.$$

Solution. We set $y = x^2 + x + 4$. Then the given equation takes the form:

$$y^2 + 8xy + 15x^2 = 0.$$

Let us solve this equation regarding it as quadratic with respect to y:

$$y_{1,2} = -4x \pm \sqrt{16x^2 - 15x^2}.$$

Thus, $y_1 = -3x$, $y_2 = -5x$. And so, the problem is reduced to solving the following collection of equations:

$$x^2 + x + 4 = -3x; \quad x^2 + x + 4 = -5x.$$

From this collection we find: $x_{1,2} = -2$, $x_{3,4} = -3 \pm \sqrt{5}$.

Example 6. Solve the equation $21x^3 + x^2 - 5x - 1 = 0$.

Solution. Equations whose left-hand side represents a polynomial with integer coefficients and the constant term equal to 1 or -1 are readily transformed to reduced equations with the aid of termwise division by x in senior power (it is easy to see that such division does not lead to a loss of roots since $x = 0$ is not a root of the equation whose constant term is different from zero) and subsequent replacement of $\frac{1}{x}$ by y. In our example we get:

$$21 + \frac{1}{x} - \frac{5}{x^2} - \frac{1}{x^3} = 0.$$

Setting $\frac{1}{x} = y$, we come to the equation $21 + y - 5y^2 - y^3 = 0$, and, further, to $y^3 + 5y^2 - y - 21 = 0$. Finding by the trial method, as in Example 3, the integer root of the equation $y_1 = -3$ and dividing the polynomial $y^3 + 5y^2 - y - 21$ by $y + 3$, we get the quadratic trinomial $y^2 + 2y - 7$ with the roots $y_{2,3} = -1 \pm 2\sqrt{2}$. Since $x = \frac{1}{y}$, we have: $x_1 = -\frac{1}{3}$, $x_{2,3} = \frac{1 \pm 2\sqrt{2}}{7}$.

Example 7. Solve the equation $4x^3 - 10x^2 + 14x - 5 = 0$.

Solution. Here, we apply another method of transforming a non-reduced equation to a reduced one (the purpose of the transformation is clear: only integers serve as rational roots of a reduced equation, and we do have a method of finding integer roots). Let us multiply both sides of the given equation by a number such that the coefficient of x^3 becomes the cube of some whole number. In our case, the number 2 may serve as such a multiplier:

$$8x^3 - 20x^2 + 28x - 10 = 0.$$

We now set $y = 2x$, then the equation takes the form:

$$y^3 - 5y^2 + 14y - 10 = 0.$$

The same as in the previous examples, we find the roots of the reduced equation: $y_1=1$, $y_{2,3}=2\pm i\sqrt{6}$. Since $x=\frac{y}{2}$, the roots of the original equation are: $x_1=\frac{1}{2}$, $x_{2,3}=\frac{2\pm i\sqrt{6}}{2}$.

Example 8. Solve the equation

$$3x^4-2x^3+4x^2-4x+12=0. \qquad (5)$$

Solution. The equation has an interesting peculiarity: the ratio of its first coefficient to the constant term is equal to the square of the ratio of the second coefficient to the last but one. Such equations are called *reciprocal*. This example will illustrate the method of solving a reciprocal equation of the fourth degree.

Dividing both sides of the equation by x^2 (this does not lead to a loss of a root since the value $x=0$ is not a root of the given equation), we get:

$$3x^2-2x+4-\frac{4}{x}+\frac{12}{x^2}=0,$$

and further

$$3\left(x^2+\frac{4}{x^2}\right)-2\left(x+\frac{2}{x}\right)+4=0. \qquad (6)$$

Setting $x+\frac{2}{x}=y$, we get: $\left(x+\frac{2}{x}\right)^2=y^2$, and therefore $x^2+\frac{4}{x^2}=y^2-4$. Replacing in Equation (6) $x+\frac{2}{x}$ by y, and $x^2+\frac{4}{x^2}$ by y^2-4, we get: $3(y^2-4)-2y+4=0$, whence we find: $y_1=2$, $y_2=-\frac{4}{3}$.

The problem is thus reduced to solving the following collection of equations:

$$x+\frac{2}{x}=2; \quad x+\frac{2}{x}=-\frac{4}{3}.$$

From this collection we find: $x_{1,2}=1\pm i$, $x_{3,4}=-\frac{2}{3}\pm\frac{i\sqrt{14}}{3}$, which are the roots of Equation (5).

Example 9. Solve the equation $x^2+\frac{9x^2}{(x+3)^2}=27.$

Solution. The left-hand side of the equation represents the sum of two squares. This suggests that we should add to both sides of the equation a function such that the left-hand side turns into a perfect square of a sum. Thus, adding $-2x\,\frac{3x}{x+3}$, we get:

$$\left(x-\frac{3x}{x+3}\right)^2=27-6\,\frac{x^2}{x+3},$$

and further

$$\left(\frac{x^2}{x+3}\right)^2+6\,\frac{x^2}{x+3}-27=0.$$

Let us now set $y=\frac{x^2}{x+3}$. Then the equation takes the form: $y^2+6y-27=0$, whence $y_1=-9$, $y_2=3$.

The problem has been reduced to solving the collection of equations

$$\frac{x^2}{x+3}=-9; \qquad \frac{x^2}{x+3}=3.$$

From the first equation we find $x_{1,2}=-\frac{9}{2}\pm\frac{i3\sqrt{3}}{2}$, from the second: $x_{3,4}=\frac{3}{2}\pm\frac{3\sqrt{5}}{2}$. All the found values satisfy the condition $x+3\neq 0$, and therefore they are the roots of the original equation.

EXERCISES

In Problems 331 through 348, solve the indicated equations by factoring

331. $x^4-1=0$. **332.** $x^6-64=0$.
333. $x^4+16=0$. **334.** $x^6+1=0$.
335. $x^3+x-2=0$. **336.** $x^3-4x^2+x+6=0$.
337. $x^3+9x^2+23x+15=0$. **338.** $(x-1)^3+(2x+3)^3=27x^3+8$.
339. $2x^4-21x^3+74x^2-105x+50=0$.
340. $x^4+5x^3+4x^2-24x-24=0$.
341. $x^5-4x^4+4x^3-x^2+4x-4=0$.
342. $x^5+4x^4-6x^3-24x^2-27x-108=0$.
343. $(x+1)(x^2+2)+(x+2)(x^2+1)=2$.
344. $3\left(x+\frac{1}{x^2}\right)-7\left(1+\frac{1}{x}\right)=0$. **345.** $\frac{(3+x)(2+x)(1+x)}{(3-x)(2-x)(1-x)}=-35$.
346. $\frac{x-2}{x-1}+\frac{x+2}{x+1}=\frac{x-4}{x-3}+\frac{x+4}{x+3}-\frac{28}{15}$.
347. $2x^4-x^3+5x^2-x+3=0$.
348. $2x^4-4x^3+13x^2-6x+15=0$.

In Problems 349 through 362, solve the given equations by introducing an auxiliary variable:

349. $x^8-15x^4-16=0$. **350.** $(x^2-5x+7)^2-(x-2)(x-3)=1$.
351. $(x^2-2x-5)^2-2(x^2-2x-3)-4=0$.
352. $\frac{x^2+1}{x}+\frac{x}{x^2+1}=2.9$. **353.** $\frac{3}{1+x+x^2}=3-x-x^2$.
354. $\frac{x^2-x}{x^2-x+1}-\frac{x^2-x+2}{x^2-x-2}=1$. **355.** $\frac{1}{x^2-3x+3}+\frac{2}{x^2-3x+4}=\frac{6}{x^2-3x+5}$.
356. $x^3-x^2-\frac{8}{x^3-x^2}=2$.
357. $x(x-1)(x-2)(x-3)=15$.
358. $(x-1)x(x+1)(x+2)=24$.
359. $(x+1)(x+2)(x+3)(x+4)=3$.

360. $(8x + 7)^2 (4x + 3)(x + 1) = 4.5$.
361. $(x - 4.5)^4 + (x - 5.5)^4 = 1$. **362.** $(x + 3)^4 + (x + 5)^4 = 16$.

In Problems 363 through 383, solve the given equations:

363. $10x^3 - 3x^2 - 2x + 1 = 0$. **364.** $4x^3 - 3x - 1 = 0$.
365. $38x^3 + 7x^2 - 8x - 1 = 0$. **366.** $4x^3 + 6x^2 + 4x + 1 = 0$.
367. $16x^3 - 28x^2 + 4x + 3 = 0$. **368.** $100x^3 - 120x^2 + 47x - 6 = 0$.
369. $6x^3 - 13x^2 + 9x - 2 = 0$. **370.** $4x^3 + 6x^2 + 5x + 69 = 0$.
371. $3x^3 - 2x^2 + x - 10 = 0$. **372.** $32x^3 - 24x^2 - 12x - 77 = 0$.
373. $4x^3 + 2x^2 - 8x + 3 = 0$.

374. $2\left(x^2+\frac{1}{x^2}\right)-7\left(x+\frac{1}{x}\right)+9=0$. **375.** $4x^2+12x+\frac{12}{x}+\frac{4}{x^2}=47$.

376. $x^2+x+x^{-1}+x^{-2}=4$. **377.** $\frac{x^2}{3}+\frac{48}{x^2}=5\left(\frac{x}{3}+\frac{4}{x}\right)$.

378. $x^4-2x^3-x^2-2x+1=0$. **379.** $x^4+x^3+4x^2+5x+25=0$.

380. $x^4+2x^3-7x^2-4x+4=0$. **381.** $16x^4+8x^3-7x^2+2x+1=0$.

382. $x^4-8x+63=0$. **383.** $\frac{x^2-6x-9}{x}=\frac{x^2-4x-9}{x^2-6x-9}$.

SEC. 9. EQUATIONS CONTAINING MODULUS OF THE VARIABLE

When solving equations containing modulus of the variable, the following methods are applied most frequently: (1) using the definition of the modulus; (2) squaring both sides of an equation; (3) subdividing into intervals.

Example 1. Solve the equation

$$|2x - 3| = 5. \tag{1}$$

Solution. First Method. Since, by hypothesis,

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geqslant 0, \\ -f(x) & \text{if } f(x) < 0, \end{cases}$$

Equation (1) is equivalent to the collection of two mixed systems:

$$\begin{cases} 2x-3\geqslant 0 \\ 2x-3=5 \end{cases}; \quad \begin{cases} 2x-3<0 \\ -(2x-3)=5. \end{cases}$$

From the first system of this collection we find $x_1 = 4$, and from the second $x_2 = -1$.

Second Method. Since both sides of Equation (1) are nonnegative, the equation is equivalent to the following: $|2x - 3|^2 = 25$. But $|f(x)|^2 = (f(x))^2$. Therefore Equation (1) is equivalent to the equation $(2x - 3)^2 = 25$, whence we get: $x_1 = 4$, $x_2 = -1$.

Example 2. Solve the equation

$$|2x - 3| = x + 1. \tag{2}$$

Solution. This equation, like the preceding one, can be solved by two methods. When solving by the first method, we get the collection of mixed systems (equivalent to Equation (2)):

$$\begin{cases} 2x-3\geqslant 0 \\ 2x-3=x+1 \end{cases}; \qquad \begin{cases} 2x-3<0 \\ -(2x-3)=x+1, \end{cases}$$

whence we find: $x_1 = 4$, $x_2 = \frac{2}{3}$.

When solving by the second method, it should be borne in mind that the expression $x + 1$ on the right-hand side of Equation (2) must be, by the sense of the equation, nonnegative: $x + 1 \geqslant 0$. On this condition, squaring both sides of the equation results in an equation equivalent to the original one. Hence, Equation (2) is equivalent to the mixed system

$$\begin{cases} x+1\geqslant 0 \\ (2x-3)^2=(x+1)^2. \end{cases}$$

Solving this system, we get: $x_1 = 4$, $x_2 = \frac{2}{3}$.

Example 3. Solve the equation

$$|\,2x-3\,| = |\,x+7\,|. \tag{3}$$

Solution. It is easy to get convinced that squaring as the method of solution (the second method) is the most advisable here. Indeed, when solving by this method, we get one equation equivalent to Equation (3): $(2x - 3)^2 = (x + 7)^2$, whence $x_1 = 10$, $x_2 = -\frac{4}{3}$.

Example 4. Solve the equation

$$|\,3-x\,| - |\,x+2\,| = 5. \tag{4}$$

Solution. In this case, the method of subdividing into intervals (the third method) is preferable.

We mark on the number line the value of x for which $3 - x = 0$ and the value of x for which $x + 2 = 0$. The number line is thereby subdivided into three intervals: $(-\infty, -2)$, $[-2, 3]$, $(3, \infty)$. We then solve Equation (4) on each of these intervals, that is, solve the collection of mixed systems equivalent to Equation (4):

$$\begin{cases} -\infty < x < -2 \\ 3-x+x+2=5 \end{cases}; \qquad \begin{cases} -2\leqslant x\leqslant 3 \\ 3-x-x-2=5 \end{cases};$$

$$\begin{cases} 3 < x < \infty \\ -3+x-x-2=5, \end{cases}$$

or

$$\begin{cases} x < -2 \\ 5 = 5 \end{cases}; \quad \begin{cases} -2 \leqslant x \leqslant 3 \\ x = -2 \end{cases}; \quad \begin{cases} x > 3 \\ -5 = 5. \end{cases}$$

The solution of the first system of this collection is the ray $(-\infty, -2)$, from the second system we find that $x = -2$, while the third system has no solution. Combining the solutions of these three systems, we get the solution of Equation (4): $(-\infty, -2]$.

Example 5. Solve the equation

$$|x - 2| + |x - 1| = x - 3. \tag{5}$$

Solution. Equation (5) is very much akin to the equation solved in the preceding example, that is, it may seem at first glance that the most suitable way of solution is applying the method of subdividing into intervals. But from Equation (5) it is clear that $x - 3 > 0$, that is, $x > 3$, and then also $x - 2 > 0$ and $x - 1 > 0$. Thus, Equation (5) is equivalent to the mixed system

$\begin{cases} x - 2 + x - 1 = x - 3, \\ x > 3 \end{cases}$ which is equivalent to the system $\begin{cases} x = 0, \\ x > 3 \end{cases}$ having no solution. Thus the equation has no roots.

EXERCISES

In Problems 384 through 417, solve the indicated equations:

384. $|x| + x^3 = 0$. 385. $(x + 1)(|x| - 1) = -0.5$.
386. $\dfrac{4x-8}{|x-2|} = x$. 387. $\dfrac{7x+4}{5} - x = \dfrac{|3x-5|}{2}$.
388. $7 - 4x = |4x - 7|$. 389. $|3x - 5| = 5 - 3x$.
390. $|x^2 - 3x + 3| = 2$. 391. $|2x - x^2 + 3| = 2$.
392. $|x^2 + x - 1| = 2x - 1$. 393. $|x^2 - x - 3| = -x - 1$.
394. $2|x^2 + 2x - 5| = x - 1$. 395. $x^2 + 3|x| + 2 = 0$.
396. $(x + 1)^2 - 2|x + 1| + 1 = 0$. 397. $x^2 + 2x - 3|x + 1| + 3 = 0$.
398. $|x| + |x + 1| = 1$. 399. $|x + 1| + |x + 2| = 2$.
400. $|x - 1| - |x - 2| = 1$. 401. $|x - 2| + |4 - x| = 3$.
402. $|x - 1| + |x - 2| = 1$. 403. $|x - 2| + |x - 3| + |2x - 8| = 9$.
404. $|2x + 1| - |3 - x| = |x - 4|$. 405. $|x - 1| + |1 - 2x| = 2|x|$.
406. $|x| - 2|x + 1| + 3|x + 2| = 0$.
407. $|x + 1| - |x| + 3|x - 1| - 2|x - 2| = |x + 2|$.
408. $|x| - 2|x + 1| + 3|x + 2| = 0$.
409. $|x| + 2|x + 1| - 3|x - 3| = 0$.
410. $|x^2 - 9| + |x - 2| = 5$. 411. $|x^2 - 1| + x + 1 = 0$.
412. $|x^2 - 4| - |9 - x^2| = 5$. 413. $|x^2 - 9| + |x^2 - 4| = 5$.
414. $|x - x^2 - 1| = |2x - 3 - x^2|$.
415. $|x^2 + 2x| - |2 - x| = |x^2 - x|$.
416. $||3 - 2x| - 1| = 2|x|$. 417. $\dfrac{|x^2 - 4x| + 3}{x^2 + |x - 5|} = 1$.

SEC. 10. SYSTEM OF RATIONAL EQUATIONS

1. Basic Concepts. Several equations in two variables x, y form a *system* if the problem is posed to find all such pairs (x, y) which satisfy each of the given equations. Each pair is called a *solution* of the system. To solve a system of equations means to find all of its solutions. The set of solutions of a system may be, in particular, empty. In such a case, we say that the system has no solution or that the equations are *incompatible*.

Several systems of equations in two variables x, y form a *collection of systems* if the problem is posed to find all such pairs (x, y) each of which satisfies at least one of the given systems. Each of the pairs is called a *solution of the collection of systems.*

The process of solving a system of equations consists, as a rule, in a subsequent passage, with the aid of certain transformations, from one system to another, more "convenient"; then to a still more "convenient" system, and so forth. If as a result of some transformations of the system

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ f_2(x, y) = g_2(x, y) \\ \cdots\cdots\cdots \\ f_n(x, y) = g_n(x, y) \end{cases} \tag{1}$$

we passed to the system

$$\begin{cases} f_1'(x, y) = g_1'(x, y) \\ f_2'(x, y) = g_2'(x, y) \\ \cdots\cdots\cdots \\ f_n'(x, y) = g_n'(x, y) \end{cases} \tag{2}$$

and if each solution of System (1) is at the same time a solution of System (2), then (2) is called a *consequence* of System (1) (or a *derived system*). A consequence may sometimes consist of only one equation. For instance, the equation $3x - 2y = 3$ is a consequence of the system

$$\begin{cases} 2x + y = 5 \\ x - 3y = -2 \end{cases}$$

(as the sum of the equations entering this system). In general, a derived system of equations may contain either less or more equations than the original system. Thus, the system

$$\begin{cases} 2x + y = 5 \\ x - 3y = -2 \\ 3x - 2y = 3 \end{cases}$$

is derived from the system

$$\begin{cases} 2x + y = 5 \\ x - 3y = -2 \end{cases}$$

which, in turn, is derived from the system

$$\begin{cases} 2x + y = 5 \\ x - 3y = -2 \\ 3x - 2y = 3. \end{cases}$$

Two systems of equations are called *equivalent* if the sets of their solutions coincide. It is clear that two systems are equivalent if and only if the systems can be derived from each other. Hence, in particular, it follows that the addition of one more equation to a system of equations yields a new system, equivalent to the original, provided that this equation is derived from the given system. And if one of the equations of a system is omitted, then the remaining equation (or a system of equations) is a consequence of the original system, or a derived system. If it is not stipulated on what set a system of rational equations is required to be solved, then the system is supposed to be solved on the set of complex numbers.

Let us formulate two theorems used for solving systems of equations.

Theorem 1. *If the equation $f_1(x, y) = g_1(x, y)$ is equivalent to (derived from) the equation $f_1'(x, y) = g_1'(x, y)$, and the equation $f_2(x, y) = g_2(x, y)$ is equivalent to (derived from) the equation $f_2'(x, y) = g_2'(x, y)$, then the systems*

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ f_2(x, y) = g_2(x, y) \end{cases} \quad \textit{and} \quad \begin{cases} f_1'(x, y) = g_1'(x, y) \\ f_2'(x, y) = g_2'(x, y) \end{cases}$$

are equivalent (the second system is derived from the first).

Theorem 2. *If the equation $f(x, y) = g(x, y)$ is derived from the equations $f_1(x, y) = g_1(x, y)$ and $f_2(x, y) = g_2(x, y)$, then the system*

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ f(x, y) = g(x, y) \end{cases} \quad \textit{or} \quad \begin{cases} f_2(x, y) = g_2(x, y) \\ f(x, y) = g(x, y) \end{cases}$$

is a consequence of the system

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ f_2(x, y) = g_2(x, y), \end{cases} \tag{3}$$

while the system

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ f_2(x, y) = g_2(x, y) \\ f(x, y) = g(x, y) \end{cases}$$

is equivalent to System (3).

In particular, the following systems are consequences of System (3):

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ f_1(x, y) \pm f_2(x, y) = g_1(x, y) \pm g_2(x, y), \end{cases} \tag{4}$$

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ f_1(x, y)\, f_2(x, y) = g_1(x, y)\, g_2(x, y), \end{cases} \tag{5}$$

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ (f_1(x, y))^2 = (g_1(x, y))^2. \end{cases} \tag{6}$$

If there are no such pairs (x, y) for which both $f_2(x, y)$ and $g_2(x, y)$ vanish simultaneously, then the equation $\frac{1}{f_2(x, y)} = \frac{1}{g_2(x, y)}$ is equivalent to the equation $f_2(x, y) = g_2(x, y)$. Then the following system is equivalent to System (3):

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ \dfrac{1}{f_2(x, y)} = \dfrac{1}{g_2(x, y)}, \end{cases}$$

the following system being, in turn, a consequence of this system

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ f_1(x, y)\, \dfrac{1}{f_2(x, y)} = g_1(x, y)\, \dfrac{1}{g_2(x, y)}. \end{cases}$$

Thus, we come to the following conclusion: *if there are no such pairs (x, y) for which both $f_2(x, y)$ and $g_2(x, y)$ vanish simultaneously, then the system*

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ \dfrac{f_1(x, y)}{f_2(x, y)} = \dfrac{g_1(x, y)}{g_2(x, y)} \end{cases} \tag{7}$$

is a consequence of System (3).

If in the process of solving a system we transformed it into a consequence of the original system, then the found solutions of the new system must be undoubtedly checked (say, by substituting

the found values of the variables into the original system). The following statements will be useful for future considerations:

1. System (4) is equivalent to System (3).
2. If there are no such pairs (x, y) for which both sides of the equation $f_1(x, y) = g_1(x, y)$ vanish simultaneously, then System (5) is equivalent to System (3).
3. System (6) is equivalent to System (3) over the field of real numbers if for any x, y from the domain of definition of System (3) the inequality $f_2(x, y)\, g_2(x, y) \geqslant 0$ is fulfilled.
4. If there are no such pairs (x, y) for which both sides of the second equation of System (3) vanish simultaneously, then System (7) is equivalent to System (3).

Let us note one more result of Theorems 1 and 2.

Theorem 3. *If the collection of equations*

$$\left[\begin{array}{l} f_{21}(x, y) = g_{21}(x, y) \\ f_{22}(x, y) = g_{22}(x, y) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f_{2k}(x, y) = g_{2k}(x, y) \end{array}\right.$$

is equivalent to the equation $f_2(x, y) = g_2(x, y)$ *or is its consequence, then the collection of systems*

$$\left[\begin{array}{l} \begin{cases} f_1(x, y) = g_1(x, y) \\ f_{21}(x, y) = g_{21}(x, y) \end{cases} \\ \begin{cases} f_1(x, y) = g_1(x, y) \\ f_{22}(x, y) = g_{22}(x, y) \end{cases} \\ \begin{cases} \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ f_1(x, y) = g_1(x, y) \\ f_{2k}(x, y) = g_{2k}(x, y) \end{cases} \end{array}\right.$$

is equivalent to System (3) (*or is its consequence*).

In particular, derived from the system

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ f_{21}(x, y) \cdot f_{22}(x, y) \cdot \ldots \cdot f_{2k}(x, y) = 0 \end{cases}$$

is the collection of systems

$$\begin{cases} f_1(x, y) = g_1(x, y) \\ f_{21}(x, y) = 0 \end{cases}; \quad \begin{cases} f_1(x, y) = g_1(x, y) \\ f_{22}(x, y) = 0 \end{cases};$$

$$\ldots; \quad \begin{cases} f_1(x, y) = g_1(x, y) \\ f_{2k}(x, y) = 0. \end{cases}$$

Example 1. Solve the system

$$\begin{cases} xy-6=\dfrac{y^3}{x} \\ xy+24=\dfrac{x^3}{y} \end{cases} \tag{8}$$

on the set of real numbers.

Solution. Multiplying together the equations of System (8), we get the system:

$$\begin{cases} xy-6=\dfrac{y^3}{x} \\ (xy+24)(xy-6)=\dfrac{y^3x^3}{xy}, \end{cases} \tag{9}$$

which is a consequence of the original system. The second equation of System (9) is reduced by rather simple transformations to the equation $xy = 8$, which is a consequence of the second equation of System (9). Then, by virtue of Theorem 1, the system

$$\begin{cases} xy-6=\dfrac{y^3}{x} \\ xy=8 \end{cases} \tag{10}$$

will be a consequence of System (9). Subtracting the first equation of System (10) from the second, we get:

$$\begin{cases} xy=8 \\ 6=8-\dfrac{y^3}{x}, \end{cases}$$

and further

$$\begin{cases} xy=8 \\ \dfrac{y^3}{x}=2. \end{cases} \tag{11}$$

By virtue of Theorem 2, System (11) is a consequence of System (10)

Multiplying together the equations of System (11), we get the system

$$\begin{cases} xy=8 \\ y^4=16, \end{cases} \tag{12}$$

which is a consequence of System (11). From the second equation of System (12) we find: $y_1 = 2$, $y_2 = -2$ (here we confine ourselves to real roots), and from the first equation: $x_1 = 4$, $x_2 = -4$.

Thus, System (12) has the solutions: (4, 2) and (—4, —2).

Check. Since System (12) is in the long run a consequence of System (8), the found solutions of the system must undergo a check which may be carried out by substituting the found solutions of System (12) into System (8). This check shows that both solutions of System (12) are at the same time solutions of System (8). Thus, the solutions of System (8) are: (4, 2), (—4, —2).

Example 2. Solve the system
$$\begin{cases} xy + xz = -4 \\ yz + yx = -1 \\ zx + zy = -9. \end{cases}$$

Solution. Adding together all the three equations, we get: $xy + xz + yz = -7$. Joining this equation to the equations of the given system, we get a system equivalent to the given (by Theorem 2):

$$\begin{cases} xy + xz + yz = -7 \\ xy + xz = -4 \\ yz + yx = -1 \\ zx + zy = -9. \end{cases}$$

We replace the second equation of this system by the difference of the first and second equations, the third one by the difference of the first and third equations, and the fourth one by the difference of the first and fourth equations. Besides, we omit the first equation and finally get the system:

$$\begin{cases} yz = -3 \\ xz = -6 \\ xy = 2, \end{cases}$$

which is equivalent to the given system, by virtue of Theorem 2 and Statement 1. Multiplying together all the three equations, we get: $(xyz)^2 = 36$. Joining this equation to the equations of the preceding system, we arrive at the equivalent system:

$$\begin{cases} (xyz)^2 = 36 \\ yz = -3 \\ xz = -6 \\ xy = 2 \end{cases}$$

(here Theorem 2 is used once again), to which, in turn, by Theorem 3, the following collection of systems is equivalent:

$$\begin{cases} xyz = 6 \\ yz = -3 \\ xz = -6 \\ xy = 2 \end{cases}; \qquad \begin{cases} xyz = -6 \\ yz = -3 \\ xz = -6 \\ xy = 2. \end{cases}$$

Let us solve the first system of this collection. Dividing the first equation of this system, in succession, by the second, third, and fourth, we get: $x = -2$, $y = -1$, $z = 3$.

Similarly, from the second system we find: $x = 2$, $y = 1$, $z = -3$. Thus, the collection of systems, and thereby the original system (which is equivalent to this collection), have the solutions: $(-2, -1, 3)$, $(2, 1, -3)$.

2. Basic Methods of Solving Systems. Let us dwell on three basic methods of solving systems of equations: (1) linear transformation of a system (or algebraic addition); (2) substitution; (3) change of variables.

The method of a *linear transformation of a system* is based on the following theorem.

Theorem 4. *If* $\Delta = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0$, *then the systems*

$$\begin{cases} f_1(x, y) = 0 \\ f_2(x, y) = 0 \end{cases} \quad \text{and} \quad \begin{cases} a_1 f_1(x, y) + a_2 f_2(x, y) = 0 \\ b_1 f_1(x, y) + b_2 f_2(x, y) = 0 \end{cases}$$

are equivalent.

In particular, if $a_1 = 1$, $a_2 = 0$, $b_1 = 1$, $b_2 = \pm 1$, then we get the system

$$\begin{cases} f_1(x, y) = 0 \\ f_1(x, y) \pm f_2(x, y) = 0, \end{cases}$$

which is equivalent to the original system (by Statement 1).

This theorem is extended to the case when the number of equations is greater than two. Say, for three equations in three variables the following theorem holds.

Theorem 4′. *If* $\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \neq 0$, *then the systems*

$$\begin{cases} f_1(x, y, z) = 0 \\ f_2(x, y, z) = 0 \\ f_3(x, y, z) = 0 \end{cases} \quad \text{and} \quad \begin{cases} a_1 f_1 + a_2 f_2 + a_3 f_3 = 0 \\ b_1 f_1 + b_2 f_2 + b_3 f_3 = 0 \\ c_1 f_1 + c_2 f_2 + c_3 f_3 = 0 \end{cases}$$

are equivalent.

The *substitution* method is based on the following theorem.

Theorem 5. *The systems of equations*

$$\begin{cases} x = F(y) \\ f(x, y) = g(x, y) \end{cases} \quad \textit{and} \quad \begin{cases} x = F(y) \\ f(F(y), y) = g(F(y), y) \end{cases}$$

are equivalent.

Thus, the following systems are equivalent:

$$\begin{cases} x = 2y - 5 \\ x^2 + y^2 = 2x + y \end{cases} \quad \text{and} \quad \begin{cases} x = 2y - 5 \\ (2y-5)^2 + y^2 = 2(2y-5) + y. \end{cases}$$

Corollary. *If the equation $\varphi(x, y) = 0$ is equivalent to the equation $x = F(y)$ (or $y = F(x)$), then the system*

$$\begin{cases} \varphi(x, y) = 0 \\ f(x, y) = g(x, y) \end{cases}$$

is equivalent to the system $\begin{cases} x = F(y) \\ f(F(y), y) = g(F(y), y) \end{cases}$ *or* $\begin{cases} y = F(x) \\ f(x, F(x)) = g(x, F(x)). \end{cases}$

For instance, the system of equations $\begin{cases} y^2 + x = 2(x-5) \\ \dfrac{y}{x} + \dfrac{x}{y} = x^2 + y^2 \end{cases}$ is equivalent to the system

$$\begin{cases} x = y^2 + 10 \\ \dfrac{y}{y^2+10} + \dfrac{y^2+10}{y} = (y^2+10)^2 + y^2. \end{cases}$$

For a system of three equations in three variables the corresponding theorem is formulated as follows:

Theorem 5′. *The system of equations*

$$\begin{cases} f_1(x, y, z) = g_1(x, y, z) \\ f_2(x, y, z) = g_2(x, y, z) \\ z = F(x, y) \end{cases}$$

is equivalent to the following:

$$\begin{cases} f_1(x, y, F(x, y)) = g_1(x, y, F(x, y)) \\ f_2(x, y, F(x, y)) = g_2(x, y, F(x, y)) \\ z = F(x, y). \end{cases}$$

The method of *change of variables* consists in the following. *If* $F_1(x, y) = f_1[\varphi_1(x, y), \varphi_2(x, y)]$ *and*

$$F_2(x, y) = f_2[\varphi_1(x, y), \varphi_2(x, y)],$$

then the system

$$\begin{cases} F_1(x, y) = 0 \\ F_2(x, y) = 0, \end{cases}$$

with the aid of new variables $\varphi_1(x, y) = u$, $\varphi_2(x, y) = v$, *can be written in the form* $\begin{cases} f_1(u, v) = 0 \\ f_2(u, v) = 0. \end{cases}$

Let $(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)$ be solutions of the last system. Then the problem is reduced to solving the collection of systems:

$$\begin{cases} \varphi_1(x, y) = u_1 \\ \varphi_2(x, y) = v_1 \end{cases}; \quad \begin{cases} \varphi_1(x, y) = u_2 \\ \varphi_2(x, y) = v_2 \end{cases}; \ldots; \quad \begin{cases} \varphi_1(x, y) = u_n \\ \varphi_2(x, y) = v_n. \end{cases}$$

The solutions of this collection are simultaneously solutions of the system:

$$\begin{cases} F_1(x, y) = 0 \\ F_2(x, y) = 0. \end{cases}$$

Consider several examples illustrating the application of these methods to solving systems of equations.

Example 3. Solve the system $\begin{cases} x^2 = 13x + 4y \\ y^2 = 4x + 13y. \end{cases}$

Solution. Subtract the second equation from the first. Then, by Theorem 4, the system

$$\begin{cases} x^2 - y^2 = (13x + 4y) - (4x + 13y) \\ y^2 = 4x + 13y \end{cases}$$

is equivalent to the original one. Consider the first equation of the obtained system. We have: $(x - y)(x + y) = 9(x - y)$, and further $(x - y)(x + y - 9) = 0$.

Finally, we arrive at the system which is equivalent to the original one (by Theorem 1):

$$\begin{cases} (x - y)(x + y - 9) = 0 \\ y^2 = 4x + 13y. \end{cases}$$

By Theorem 3, this system is equivalent to the collection of systems:

$$\begin{cases} x - y = 0 \\ y^2 = 4x + 13y \end{cases}; \qquad \begin{cases} x + y - 9 = 0 \\ y^2 = 4x + 13y. \end{cases}$$

We solve each of these systems by the substitution method. The first system is transformed as follows:

$$\begin{cases} x = y \\ y^2 = 4y + 13y, \end{cases}$$

whence we find:

$$\begin{cases} x_1 = 0 \\ y_1 = 0 \end{cases} \quad \begin{cases} x_2 = 17 \\ y_2 = 17. \end{cases}$$

The second system of the collection is transformed as follows:

$$\begin{cases} x = 9 - y \\ y^2 = 4(9 - y) + 13y. \end{cases}$$

From the equation $y^2 = 4(9 - y) + 13y$ we find: $y_3 = 12$, $y_4 = -3$, and, further, from the relationship $x = 9 - y$ we get: $x_3 = -3$, $x_4 = 12$.

As a result, we have found the four solutions: (0, 0), (17, 17), (—3, 12), (12, —3).

Check. Since in the process of solving the given system only equivalent transformations were carried out, the found solutions are just solutions of the given system.

Example 4. Solve the system of equations

$$\begin{cases} x + y + z = 2 \\ 2x + 3y + z = 1 \\ x^2 + (y + 2)^2 + (z - 1)^2 = 9. \end{cases}$$

Solution. Let us apply the substitution method. We have:

$$\begin{cases} x = 2 - y - z \\ 2(2 - y - z) + 3y + z = 1 \\ (2 - y - z)^2 + (y + 2)^2 + (z - 1)^2 = 9, \end{cases}$$

and further

$$\begin{cases} x = 2 - y - z \\ y - z = -3 \\ y^2 + z^2 + yz - 3z = 0. \end{cases}$$

The last two equations of the obtained system, in turn, form a system of two equations in two variables. Let us solve this system using the substitution method. We have:

$$\begin{cases} y = z - 3 \\ (z - 3)^2 + z^2 + (z - 3)z - 3z = 0, \end{cases}$$

that is,

$$\begin{cases} y = z - 3 \\ z^2 - 4z + 3 = 0. \end{cases}$$

From the last equation we find: $z_1 = 1$, $z_2 = 3$. From the equation $y = z - 3$ we get: $y_1 = -2$, $y_2 = 0$, and from the equation $x = 2 - y - z$ we find: $x_1 = 3$, $x_2 = -1$.

Thus, we have obtained the following solutions: (3, —2, 1), (—1, 0, 3).

Example 5. Solve the system of equations $\begin{cases} xy + z^2 = 2 \\ yz + x^2 = 2 \\ zx + y^2 = 2. \end{cases}$

Solution. Let us replace the first equation by the difference of the first and second equations, the second one by the difference of the second and third equations, leaving the third one unchanged. Then we get the system:

$$\begin{cases} xy - yz + z^2 - x^2 = 0 \\ yz - xz + x^2 - y^2 = 0 \\ xz + y^2 = 2, \end{cases}$$

that is, the system

$$\begin{cases} (z - x)(z + x) - y(z - x) = 0 \\ (x - y)(x + y) - z(x - y) = 0 \\ xz + y^2 = 2, \end{cases}$$

which is equivalent to the given system (by Theorem 4). We further have:

$$\begin{cases} (z - x)(z + x - y) = 0 \\ (x - y)(x + y - z) = 0 \\ xz + y^2 = 2. \end{cases}$$

By Theorem 3, the following collection of systems is equivalent to this system:

$$\begin{cases} z - x = 0 \\ x - y = 0 \\ xz + y^2 = 2 \end{cases}; \quad \begin{cases} z - x = 0 \\ x + y - z = 0; \\ xz + y^2 = 2 \end{cases} \quad \begin{cases} z + x - y = 0 \\ x - y = 0 \\ xz + y^2 = 2 \end{cases}; \quad \begin{cases} z + x - y = 0 \\ x + y - z = 0 \\ xz + y^2 = 2. \end{cases}$$

We are going to solve the systems of this collection using the substitution method. From the first system we find: (1, 1, 1),

$(-1, -1, -1)$; from the second: $(\sqrt{2}, 0, \sqrt{2})$, $(-\sqrt{2}, 0, -\sqrt{2})$; from the third: $(\sqrt{2}, \sqrt{2}, 0)$, $(-\sqrt{2}, -\sqrt{2}, 0)$; from the fourth: $(0, \sqrt{2}, \sqrt{2})$, $(0, -\sqrt{2}, -\sqrt{2})$.

Check. In the process of solving all transformations are equivalent, therefore all the found solutions are just solutions of the given system of equations.

3. Homogeneous Systems. A system of two equations in two variables of the form

$$\begin{cases} a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \ldots + a_{n-1}xy^{n-1} + a_ny^n = c \\ b_0x^n + b_1x^{n-1}y + b_2x^{n-2}y^2 + \ldots + b_{n-1}xy^{n-1} + b_ny^n = d \end{cases}$$

is called *homogeneous* (the left-hand sides of both equations are homogeneous polynomials of degree n in two variables). Homogeneous systems are solved using the combination of two methods: linear transformation and introduction of new variables.

Example 6. Find real solutions of the system

$$\begin{cases} 3x^2 + xy - 2y^2 = 0 \\ 2x^2 - 3xy + y^2 = -1. \end{cases}$$

Solution. The first equation of the system is homogeneous (we call so equations of the form $f(x, y) = 0$, where $f(x, y)$ is a homogeneous polynomial). Note that if we set $y = 0$, then from the equation $3x^2 + xy - 2y^2 = 0$ we find: $x = 0$. But the pair $(0, 0)$ does not satisfy the second equation of the system, therefore $y \neq 0$, and, consequently, both sides of the homogeneous equation $3x^2 + xy - 2y^2 = 0$ may be divided by y^2 (this does not lead to a loss of roots).

We get: $\frac{3x^2}{y^2} + \frac{xy}{y^2} - \frac{2y^2}{y^2} = \frac{0}{y^2}$, and further $3\left(\frac{x}{y}\right)^2 + \frac{x}{y} - 2 = 0$, whence we find that $\frac{x}{y} = -1$ or $\frac{x}{y} = \frac{2}{3}$, that is, $x = -y$ or $x = \frac{2}{3}y$.

Now the problem is reduced to solving the collection of systems of equations

$$\begin{cases} x = -y \\ 2x^2 - 3xy + y^2 = -1 \end{cases}; \quad \begin{cases} x = 2y/3 \\ 2x^2 - 3xy + y^2 = -1. \end{cases}$$

The first of these systems is incompatible, and the second has two solutions: $(2, 3)$, $(-2, -3)$. These are just solutions of the given system.

Example 7. Solve the system $\begin{cases} 3x^2-8xy+4y^2=0 \\ 5x^2-7xy-6y^2=0. \end{cases}$

Solution. Note first of all that the pair (0, 0) satisfies this system. Let now $y \neq 0$. Dividing by y^2 both sides of each homogeneous second-degree equations forming the given system, we get

$$\begin{cases} 3\left(\frac{x}{y}\right)^2-8\left(\frac{x}{y}\right)+4=0 \\ 5\left(\frac{x}{y}\right)^2-7\left(\frac{x}{y}\right)-6=0, \end{cases}$$

whence we find

$$\begin{cases} \frac{x}{y}=2;\ \frac{x}{y}=\frac{2}{3} \\ \frac{x}{y}=2;\ \frac{x}{y}=-\frac{3}{5}. \end{cases}$$

Hence, $\frac{x}{y}=2$.

Let us set $y=t$, then $x=2t$. Note that for $t=0$ and $x=0$, we get $y=0$. Thus, the solutions of the given system are pairs of the form $(2t, t)$, where $t \in \boldsymbol{R}$.

Example 8. Solve the system of equations

$$\begin{cases} 3x^2-2xy=160 \\ x^2-3xy-2y^2=8. \end{cases} \tag{13}$$

Solution. Let us multiply both sides of the second equation by 20 and subtract the obtained equation from the first equation of the system:

$$\begin{array}{r} 3x^2\ -2xy\qquad\quad =160 \\ -\ \underline{20x^2-60xy-40y^2=160} \\ -17x^2+58xy+40y^2=0\quad. \end{array}$$

We have obtained the system equivalent to System (13):

$$\begin{cases} 3x^2-2xy=160 \\ 17x^2-58xy-40y^2=0. \end{cases} \tag{14}$$

Consider the homogeneous equation

$$17x^2-58xy-40y^2=0. \tag{15}$$

If $y=0$, then from this equation we get $x=0$. But the pair (0, 0) does not satisfy the original system. Hence, $y \neq 0$ and, therefore, dividing both sides of Equation (15) by y^2, we get the equation

which is equivalent to (15):

$$17\left(\frac{x}{y}\right)^2-58\left(\frac{x}{y}\right)-40=0.$$

Setting $u=\frac{x}{y}$, we get the quadratic equation

$$17u^2-58u-40=0,$$

whose roots are: $u_1=4$, $u_2=-\frac{10}{17}$. Hence Equation (15) is equivalent to the collection of equations: $\frac{x}{y}=4$; $\frac{x}{y}=-\frac{10}{17}$, and, accordingly, System (14) is equivalent to the collection of systems:

$$\begin{cases}\frac{x}{y}=4\\ 3x^2-2xy=160\end{cases};\qquad \begin{cases}\frac{x}{y}=-\frac{10}{17}\\ 3x^2-2xy=160.\end{cases}$$

Applying the substitution method to each of these systems, we find the following solutions: $(8, 2)$, $(-8, -2)$, $\left(5, -\frac{17}{2}\right)$, $\left(-5, \frac{17}{2}\right)$.

Since in the process of solving the given system we used only equivalent transformations, the found roots are also solutions of the original system.

Example 9. Find the real solutions of the system

$$\begin{cases}x^3+y^3=1\\ x^2y+2xy^2+y^3=2.\end{cases}\tag{16}$$

Solution. Multiplying the first equation by 2 and subtracting the second equation from it we have:

$$2x^3-x^2y-2xy^2+y^3=0.$$

We get the system:

$$\begin{cases}2x^3-x^2y-2xy^2+y^3=0\\ x^3+y^3=1,\end{cases}\tag{17}$$

which is equivalent to the given system.

Consider the equation $2x^3-x^2y-2xy^2+y^3=0$.

As in the preceding example, we might divide both sides of the equation by y^2. But in the present case it is simpler to factorize the left-hand side:

$$x^2(2x-y)-y^2(2x-y)=0,$$

and further

$$(2x-y)(x-y)(x+y)=0.$$

Hence, System (17) is equivalent to the following collection:

$$\begin{cases} 2x-y=0 \\ x^3+y^3=1 \end{cases}; \quad \begin{cases} x-y=0 \\ x^3+y^3=1 \end{cases}; \quad \begin{cases} x+y=0 \\ x^3+y^3=1. \end{cases}$$

Applying the substitution method to each of these systems, we find the solutions of System (16):

$$\left(\frac{\sqrt[3]{3}}{3}, \frac{2\sqrt[3]{3}}{3}\right), \left(\frac{\sqrt[3]{4}}{2}, \frac{\sqrt[3]{4}}{2}\right).$$

Example 10. Find the real solutions of the system of equations

$$\begin{cases} x^4+x^2y^2+y^4=91 \\ x^2-xy+y^2=7. \end{cases} \tag{18}$$

Solution. This system is not homogeneous. To make it homogeneous, we square both sides of the second equation. We get the system:

$$\begin{cases} x^4+x^2y^2+y^4=91 \\ (x^2-xy+y^2)^2=49, \end{cases}$$

which is a consequence of the original system. Further, we have:

$$\begin{cases} x^4+x^2y^2+y^4=91 \\ x^4+x^2y^2+y^4-2x^3y+2x^2y^2-2xy^3=49. \end{cases}$$

Replacing the second equation of this system by the difference of the first and second equations, we get:

$$\begin{cases} x^4+x^2y^2+y^4=91 \\ x^3y-x^2y^2+xy^3=21. \end{cases} \tag{19}$$

Multiplying the first equation by 3 and subtracting from it the second equation multiplied by 13, we get:

$$3x^4-13x^3y+16x^2y^2-13xy^3+3y^4=0. \tag{20}$$

If $y=0$, then $x=0$. But the pair (0, 0) does not satisfy System (18). If $y \neq 0$, then the division of both sides of Equation (20) by y^4 leads to the equation

$$3\left(\frac{x}{y}\right)^4-13\left(\frac{x}{y}\right)^3+16\left(\frac{x}{y}\right)^2-13\left(\frac{x}{y}\right)+3=0,$$

which is equivalent to Equation (20).

Setting $u=\frac{x}{y}$, we get the equation

$$3u^4-13u^3+16u^2-13u+3=0.$$

Dividing both sides of this equation by u^2, we get:

$$3u^2-13u+16-\frac{13}{u}+\frac{3}{u^2}=0,$$

and further

$$3\left(u^2+\frac{1}{u^2}\right)-13\left(u+\frac{1}{u}\right)+16=0.$$

Let us set $v=u+\frac{1}{u}$, then $u^2+\frac{1}{u^2}=v^2-2$, and we have:

$$3(v^2-2)-13v+16=0 \quad \text{or} \quad 3v^2-13v+10=0,$$

whence $v_1=1$, $v_2=\frac{10}{3}$.

Let us now solve the collection of equations: $u+\frac{1}{u}=1$; $u+\frac{1}{u}=\frac{10}{3}$.

The first equation of the collection has no real solutions; from the second equation we find: $u_1=3$, $u_2=\frac{1}{3}$. Thus, Equation (20) is equivalent to the collection of equations: $\frac{x}{y}=3$; $\frac{x}{y}=\frac{1}{3}$, and System (19) to the collection of systems:

$$\begin{cases}\frac{x}{y}=3\\ x^4+x^2y^2+y^4=91\end{cases}; \qquad \begin{cases}\frac{x}{y}=\frac{1}{3}\\ x^4+x^2y^2+y^4=91.\end{cases} \tag{21}$$

This collection has the following solutions: (3, 1), (1, 3), (—3, —1), (—1, —3).

Check. In the process of solving all the transformations, except the first one, led to equivalent systems. Substituting the found solutions into System (18), we get convinced that it is satisfied by all the four solutions of Collection (21).

4. **Symmetric Systems.** Let us recall the fundamentals of symmetric expressions. The expression $F(x, y)$ is said to be *symmetric* if it remains unchanged when the variables x and y are interchanged. Given below are examples of symmetric expressions:

$$F(x, y)=x^2+3xy+y^2,$$
$$F(x, y)=\sqrt{x+y}+2xy+\frac{1}{x}+\frac{1}{y}.$$

The basic symmetric polynomials in two variables are: $x+y$ and xy. The rest of symmetric polynomials in two variables can be expressed in terms of the basic ones. Setting for brevity $u=x+y$,

$v = xy$, we get, for instance:

$$\begin{aligned}
x^2 + y^2 &= (x+y)^2 - 2xy = u^2 - 2v,\\
x^3 + y^3 &= (x+y)(x^2 - xy + y^2) = u(u^2 - 3v) = u^3 - 3uv,\\
x^4 + y^4 &= (x^2 + y^2)^2 - 2x^2y^2 = (u^2 - 2v)^2 - 2v^2\\
&= u^4 - 4u^2v + 2v^2,\\
x^5 + y^5 &= (x^2 + y^2)(x^3 + y^3) - x^2y^2(x+y)\\
&= (u^2 - 2v)(u^3 - 3uv) - v^2u = u^5 - 5u^3v + 5uv^2,\\
x^2 + xy + y^2 &= (x^2 + 2xy + y^2) - xy = u^2 - v \text{ and so forth.}
\end{aligned}$$

A system all equations of which are symmetric is called *symmetric.* It can be solved by the method of change of variables, by choosing the basic symmetric polynomials as new variables.

Example 11. Solve the system of equations

$$\begin{cases} x^3 + x^3y^3 + y^3 = 17\\ x + xy + y = 5. \end{cases}$$

Solution. Let us set $\begin{cases} x + y = u\\ xy = v. \end{cases}$ Since $x^3 + y^3 = u^3 - 3uv$, the given system is reduced to the following:

$$\begin{cases} u^3 - 3uv + v^3 = 17\\ u + v = 5. \end{cases}$$

From this system we find:

$$\begin{cases} u_1 = 3\\ v_1 = 2 \end{cases}; \quad \begin{cases} u_2 = 2\\ v_2 = 3. \end{cases}$$

It now remains to solve the following collection of systems:

$$\begin{cases} x + y = 3\\ xy = 2 \end{cases}; \quad \begin{cases} x + y = 2\\ xy = 3. \end{cases}$$

The solutions of this collection and, hence, of the original system are: $(1, 2)$, $(2, 1)$, $(1 + i\sqrt{2}, 1 - i\sqrt{2})$, $(1 - i\sqrt{2}, 1 + i\sqrt{2})$.

Remark. Let us return to the system considered in Example 10:

$$\begin{cases} x^4 + x^2y^2 + y^4 = 91\\ x^2 - xy + y^2 = 7. \end{cases}$$

This system is symmetric, and therefore, much like the preceding one, can be reduced to a simpler form by using new variables:

$$\begin{cases} x + y = u\\ xy = v. \end{cases}$$

We get: $\begin{cases} ((u^2 - 2v)^2 - 2v^2) + v^2 = 91 \\ (u^2 - 2v) - v = 7, \end{cases}$

and further $\begin{cases} (u^2 - 2v)^2 - v^2 = 91 \\ u^2 - 3v = 7. \end{cases}$

From the second equation of this system we find: $u^2 = 3v + 7$. With the aid of this substitution, the first equation of the system is transformed to $(3v + 7 - 2v)^2 - v^2 = 91$, whence we find: $v = 3$.

From the equation $u^2 = 3v + 7$ we find: $u_{1,2} = \pm 4$. Thus, the system has two solutions:

$$\begin{cases} u_1 = 4 \\ v_1 = 3 \end{cases}; \quad \begin{cases} u_2 = -4 \\ v_2 = 3. \end{cases}$$

Hence, the original system is equivalent to the collection of systems:

$$\begin{cases} x+y=4 \\ xy=3 \end{cases}; \quad \begin{cases} x+y=-4 \\ xy=3. \end{cases}$$

This collection yields the same solutions as were obtained in Example 10.

EXERCISES

In Problems 418 through 452, solve the given systems of equations:

418. $\begin{cases} x^2+y^2+6x+2y=0 \\ x+y+8=0. \end{cases}$

419. $\begin{cases} x-y=1 \\ x^2+y^2=41. \end{cases}$ 420. $\begin{cases} 2x^2-3y=23 \\ 3y^2-8x=59. \end{cases}$

421. $\begin{cases} 5x^2+14y=19 \\ 7y^2+10x=17. \end{cases}$ 422. $\begin{cases} x^2(x+y)=80 \\ x^2(2x-3y)=80. \end{cases}$

423. $\begin{cases} x-y=2 \\ x^3-y^3=8. \end{cases}$ 424. $\begin{cases} x+y+z=3 \\ x+2y-z=2 \\ x+yz+zx=3. \end{cases}$

425. $\begin{cases} x^2+3y^2-xz=6 \\ 2x-y+3z=11 \\ x+2y-2z=1. \end{cases}$ 426. $\begin{cases} 9x^2+y^2=13 \\ xy=2. \end{cases}$

427. $\begin{cases} x^2+y^2-2x+3y-9=0 \\ 2x^2+2y^2+x-5y-1=0. \end{cases}$ 428. $\begin{cases} x^2-xy-y^2+x-2y=-2 \\ 3xy-5y^2+3x-6y=-5. \end{cases}$

429. $\begin{cases} x+yz=2 \\ y+zx=2 \\ z+xy=2. \end{cases}$ 430. $\begin{cases} x-y=\dfrac{1}{4}xy \\ x^2+y^2=\dfrac{5}{2}xy. \end{cases}$ 431. $\begin{cases} \dfrac{1}{x+y}+\dfrac{1}{x-y}=2 \\ \dfrac{3}{x+y}+\dfrac{4}{x-y}=7. \end{cases}$

432. $\begin{cases} \dfrac{4}{x+y}+\dfrac{4}{x-y}=3 \\ (x+y)^2+(x-y)^2=20. \end{cases}$

433. $\begin{cases} \dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{5}{2} \\ x^2+y^2=20. \end{cases}$

434. $\begin{cases} (x+y)^2+2x=35-2y \\ (x-y)^2-2y=3-2x. \end{cases}$

435. $\begin{cases} 12(x+y)^2+x=2.5-y \\ 6(x-y)^2+x=0.125+y. \end{cases}$

436. $\begin{cases} y^2(x^2-3)+xy+1=0 \\ y^2(3x^2-6)+xy+2=0. \end{cases}$

437. $\begin{cases} \dfrac{3}{x^2+y^2-1}+\dfrac{2y}{x}=1 \\ x^2+y^2+\dfrac{4x}{y}=22. \end{cases}$

438. $\begin{cases} 6x^2+xy-2y^2=0 \\ 3x^2-xy-2y^2=0. \end{cases}$

439. $\begin{cases} 56x^2-xy-y^2=0 \\ 14x^2+19xy-3y^2=0. \end{cases}$

440. $\begin{cases} 4x^2-3xy-y^2=0 \\ 32x^2-36xy+9y^2=6. \end{cases}$

441. $\begin{cases} 15x^2+xy-2y^2=0 \\ 7x^2-4xy-3y^2=-32. \end{cases}$

442. $\begin{cases} x^2+xy+4y^2=6 \\ 3x^2+8y^2=14. \end{cases}$

443. $\begin{cases} x^2-3xy+y^2=-1 \\ 3x^2-xy+3y^2=13. \end{cases}$

444. $\begin{cases} 5x^2-6xy+5y^2=29 \\ 7x^2-8xy+7y^2=43. \end{cases}$

445. $\begin{cases} x^3+y^3=35 \\ x^2y+xy^2=30. \end{cases}$

446. $\begin{cases} x^3-y^3=19(x-y) \\ x^3+y^3=7(x+y). \end{cases}$

447. $\begin{cases} x^4-y^4=15 \\ x^3y-xy^3=6. \end{cases}$

448. $\begin{cases} x^4+6x^2y^2+y^4=136 \\ x^3y+xy^3=30 \end{cases}$ (find only real solutions).

449. $\begin{cases} x^2+xy+y^2=19(x-y)^2 \\ x^2-xy+y^2=7(x-y). \end{cases}$

450. $\begin{cases} x^2+4xy-2y^2=5(x+y) \\ 5x^2-xy-y^2=7(x+y). \end{cases}$

451. $\begin{cases} x^2+y^2=34 \\ x+y+xy=23. \end{cases}$

452. $\begin{cases} x+y+x^2+y^2=18 \\ xy+x^2+y^2=19. \end{cases}$

In Problems 453 through 479, find the real solutions of the given systems of equations:

453. $\begin{cases} x^3+y^3=19 \\ (xy+8)(x+y)=2. \end{cases}$

454. $\begin{cases} \dfrac{x^2}{y}+\dfrac{y^2}{x}=12 \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{3}. \end{cases}$

455. $\begin{cases} xy(x+y)=20 \\ \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{5}{4}. \end{cases}$

456. $\begin{cases} x^2+y^2=7+xy \\ x^3+y^3=6xy-1. \end{cases}$

457. $\begin{cases} x+y=5 \\ x^4+y^4=97. \end{cases}$

458. $\begin{cases} x^4-x^2y^2+y^4=601 \\ x^2-xy+y^2=21. \end{cases}$

459. $\begin{cases} x^5+y^5=33 \\ x+y=3. \end{cases}$

460. $\begin{cases} x^3y+xy^3=\dfrac{10}{9}(x+y)^2 \\ x^4y+xy^4=\dfrac{2}{3}(x+y)^3. \end{cases}$

461. $\begin{cases} \dfrac{x^5+y^5}{x^3+y^3}=\dfrac{31}{7} \\ x^2+xy+y^2=3. \end{cases}$

462. $\begin{cases} x^3-y^3=26 \\ x^4-y^4=20\,(x+y). \end{cases}$

463. $\begin{cases} x^2-yz=3 \\ y^2-zx=5 \\ z^2-xy=-1. \end{cases}$

464. $\begin{cases} x^2+xy+y^2=7 \\ y^2+yz+z^2=3 \\ z^2+zx+x^2=1. \end{cases}$

465. $\begin{cases} 2x^2+y^2+z^2=9+yz \\ x^2+2y^2+z^2=6+zx \\ x^2+y^2+2z^2=3+xy. \end{cases}$

466. $\begin{cases} x^2y=x+y-z \\ z^2x=x-y+z \\ y^2z=y-x+z. \end{cases}$

467. $\begin{cases} x+y+z=6 \\ x\,(y+z)=5 \\ y\,(x+z)=8. \end{cases}$

468. $\begin{cases} x-y+z=6 \\ x^2+y^2+z^2=14 \\ x^3-y^3+z^3=36. \end{cases}$

469. $\begin{cases} y+z=xyz \\ z+x=xyz \\ x+y=xyz. \end{cases}$

470. $\begin{cases} \dfrac{yz}{x}=\dfrac{10}{3} \\ \dfrac{zx}{y}=\dfrac{15}{2} \\ \dfrac{xy}{z}=\dfrac{6}{5}. \end{cases}$

471. $\begin{cases} x+y+z=13 \\ x^2+y^2+z^2=91 \\ y^2=xz. \end{cases}$

472. $\begin{cases} 2x+y+z=6 \\ 3x+2y+z=7 \\ (x-1)^3+(y+2)^3+(z-3)^3=7 \end{cases}$

473. $\begin{cases} \dfrac{3xy}{x+y}=2 \\ \dfrac{4xz}{x+z}=3 \\ \dfrac{5yz}{y+z}=6. \end{cases}$

474. $\begin{cases} \dfrac{x}{y}+\dfrac{y}{z}+\dfrac{z}{x}=3 \\ \dfrac{y}{x}+\dfrac{z}{y}+\dfrac{x}{z}=3 \\ x+y+z=3. \end{cases}$

475. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=3 \\ \dfrac{1}{xy}+\dfrac{1}{yz}+\dfrac{1}{zx}=3 \\ \dfrac{1}{xyz}=1. \end{cases}$

476. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{7}{2} \\ x+y+z=\dfrac{7}{2} \\ xyz=1. \end{cases}$

477. $\begin{cases} x^2+y^2=z^2 \\ xy+yz+zx=47 \\ (z-x)\,(z-y)=2. \end{cases}$

478. $\begin{cases} x+y=3z \\ x^2+y^2=5z \\ x^3+y^3=9z. \end{cases}$

479. $\begin{cases} x-z=y^2 \\ x^2-z^2=3y^4 \\ y^3+3y+x+z=26. \end{cases}$

SEC. 11. PROBLEMS ON SETTING UP EQUATIONS AND SYSTEMS OF EQUATIONS

The solution of textual problems by setting up equations is usually realized in four subsequent steps: (1) denoting the unknown quantities of a problem by the letters x, y, z, . . .; (2) setting up a system of equations (or one equation) using the introduced variables and

quantities known from the conditions of the problem; (3) solving the obtained system of equations (or one equation); (4) choosing the solutions according to the sense of the problem.

1. Problems on Numerical Relations. When solving such problems, we use the following facts:

1. If we add to the right an n-digit number y to a natural number x, then we get the number $10^n x + y$.

2. If a and b are natural numbers, where $a > b$ and a is not multiple of b, then there is only one pair of natural numbers q and r such that $a = bq + r$, where $r < b$ (a dividend, b divisor, q quotient, r remainder).

Example 1. Find the two-digit number if it is known that its units digit exceeds by 2 its tens digit and that the product of the desired number and the sum of its digits is equal to 144.

Solution. Let x be the tens digit, y the units digit of the desired number. Then the number itself has the form $10x + y$. It follows from the conditions of the problem that, firstly, $y - x = 2$ and, secondly, $(10x + y)(x + y) = 144$. Thus, we obtain the system of equations

$$\begin{cases} y - x = 2 \\ (10x + y)(x + y) = 144. \end{cases}$$

This system has two solutions: (2, 4) and $\left(-3\frac{2}{11}, -1\frac{2}{11}\right)$. The second pair does not satisfy the conditions of the problem. Hence, the sought-for number is equal to 24.

Example 2. Find two two-digit numbers A and B if the following is known. If the number B and then the digit 0 are annexed to the number A on its right, and the resulting five-digit number is divided by the square of the number B, then 39 is obtained as the quotient and 575 as the remainder. Let the number B be annexed to the number A on its right. Further we subtract from the resulting four-digit number another four-digit number which is obtained by annexing the number B to the left of the number A. The difference is 1287.

Solution. Annexing the digit 0 to the right of the number B, we get the number $10B$. Annexing this three-digit number to the number A, we get $1000A + 10B$.

By the hypothesis, the five-digit number $1000A + 10B$ is the dividend, B^2 the divisor, 39 the quotient, 575 the remainder, that is, $1000A + 10B = 39B^2 + 575$.

Further, if the two-digit number B is annexed to the number A on its right, then the number $100A + B$ is obtained. And if the two-digit number A is annexed to the number B on its right, then the number $100B + A$ is obtained. By the hypothesis, $(100A + B) - (100B + A) = 1287$.

Thus, we have obtained the system of equations

$$\begin{cases} 1000A + 10B = 39B^2 + 575 \\ (100A + B) - (100B + A) = 1287. \end{cases}$$

Solving this system, we find:

$$\begin{cases} A_1 = 48 \\ B_1 = 35 \end{cases}; \quad \begin{cases} A_2 = \frac{152}{39} \\ B_2 = -\frac{355}{39}. \end{cases}$$

Obviously, the second pair does not satisfy the conditions of the problem. The sought-for numbers are 48 and 35.

2. Problems on Progressions. A sequence of numbers (a_n) is called an *arithmetic progression* if there is a number d such that for any $n \in N$ the equality $a_{n+1} = a_n + d$ is fulfilled; the number d is the *common difference*, or simply the *difference*. The sequence (b_n) in which $b_1 \neq 0$ is termed a *geometric progression* if there is a number $q \neq 0$ such that for any $n \in N$ the equality $b_{n+1} = b_n \cdot q$ is fulfilled; the number q is the *common ratio*, or simply the *ratio*.

The basic properties of the arithmetic progression:

(1) $a_n = a_1 + d(n-1)$.

(2) $S_n = \frac{a_1 + a_n}{2} \cdot n$, *where* $S_n = a_1 + a_2 + \ldots + a_n$.

(3) *A sequence* (a_n) *is an arithmetic progression if and only if for any* $n \in N$ *the equality* $a_{n+1} = \frac{a_n + a_{n+2}}{2}$ *is fulfilled* (*the characteristic property of an arithmetic progression*).

The basic properties of the geometric progression:

(1) $b_n = b_1 q^{n-1}$.

(2) $S_n = \frac{b_1(1-q^n)}{1-q}$, *where* $S_n = b_1 + b_2 + b_3 + \ldots + b_n$, $q \neq 1$.

(3) *A sequence* (b_n) *is a geometric progression if and only if for any* $n \in N$ *the equality* $|b_{n+1}| = \sqrt{b_n b_{n+2}}$ *is fulfilled* (*the characteristic property of a geometric progression*).

In practice, it is more convenient to use the equivalent equality $b_{n+1}^2 = b_n b_{n+2}$ instead of the equality $|b_{n+1}| = \sqrt{b_n b_{n+2}}$.

(4) *If a geometric progression is infinitely decreasing, that is,* $|q| < 1$, *then* $S = \frac{b_1}{1-q}$, *where* $S = \sum_{n=1}^{\infty} b_n$.

Problems on numerical relations involving progressions are reduced, as a rule, to solving systems of equations.

Example 3. Find the fifth term of an infinitely decreasing geometric progression if its sum is known to be equal to 9, and the sum of the squares of all of its terms to be equal to 40.5.

Solution. By the hypothesis, $S=9$, that is, $\frac{b_1}{1-q}=9$. Consider the series $b_1^2+b_2^2+b_3^2+\ldots+b_n^2+\ldots$. We know that its sum is equal to 40.5. Note that the terms of this series form a geometric progression with the first term b_1^2 and the ratio q^2. Hence the sum of this progression is equal to $\frac{b_1^2}{1-q^2}$. As the final result, we may write the system of equations $\begin{cases} \frac{b_1}{1-q}=9 \\ \frac{b_1^2}{1-q^2}=40.5. \end{cases}$ Solving this system, we get: $b_1=6$, $q=\frac{1}{3}$.

Thus, $b_5=b_1q^4=6\left(\frac{1}{3}\right)^4=\frac{2}{27}$.

Example 4. Three numbers form a geometric progression. If 4 is subtracted from the third number, then the numbers form an arithmetic progression. And if 1 is subtracted both from the second and third terms of the obtained arithmetic progression, then a geometric progression is obtained once again. Find these numbers.

Solution. Let x, y, z denote the sought-for numbers. Since they form a geometric progression (or more precisely, they are successive terms of a geometric progression), we may use its characteristic property and get: $y^2=xz$. Further, since the numbers x, y, $(z-4)$ form an arithmetic progression, taking advantage of its characteristic property, we get: $y=\frac{x+(z-4)}{2}$. Finally, since the numbers x, $(y-1)$, $(z-5)$ form a geometric progression, we have $(y-1)^2=x(z-5)$.

As the result, we get the system of equations: $\begin{cases} y^2=xz \\ x+z-4=2y \\ (y-1)^2=x(z-5). \end{cases}$

Solving this system, we get: $(1, 3, 9)$, $\left(\frac{1}{9}, \frac{7}{9}, \frac{49}{9}\right)$.

These values of x, y, z satisfy the conditions of the problem. Thus, the desired numbers are 1, 3 and 9 or $\frac{1}{9}$, $\frac{7}{9}$ and $\frac{49}{9}$.

Example 5. Find the three-digit number whose digits form an arithmetic progression and which is divisible by 45.

Solution. Let x be the hundreds digit, y the tens digit, and z the units digit of the sought-for number. Since the numbers x, y, z form an arithmetic progression, we have: $y=\frac{x+z}{2}$.

By the hypothesis, the sought-for number is divisible by 45, that is, both by 5 and 9. Hence, the number ends in either the digit 0 or in 5, and the sum of the digits of the sought-for number is divisible by 9. Thus, we have arrived at the collection of two systems:

$$\begin{cases} z=0 \\ y=\dfrac{x+z}{2} \\ x+y+z=9k \end{cases};\qquad \begin{cases} z=5 \\ y=\dfrac{x+z}{2} \\ x+y+z=9k. \end{cases}$$

From the first system we find:

$$\begin{cases} x=2y \\ x+y=9k. \end{cases}$$

When trying all possible values for y from 1 through 9, we see that the last system is satisfied only by the pair (6, 3).

From the second system we find: $\begin{cases} 2y = x+5 \\ x+y+5=9k. \end{cases}$

Similarly, when trying all possible values for y from 1 through 9, we get convinced that this system is satisfied by the pairs (1, 3) and (7, 6).

Thus, the conditions of the problem are satisfied by the numbers: 630, 135, 765.

3. Problems on Motion. When solving such problems, we assume the following:

1. Motion is uniform unless otherwise stated.
2. Velocity is a positive quantity.
3. Turns of moving bodies and changes in conditions of motion occur instantaneously.
4. If a body with proper speed x moves along a river whose rate of flow is y, then the speed of the body with the stream is equal to $(x+y)$, against the stream to $(x-y)$.

Example 6. A tributary flows into a river. A motor-boat puts out from the point A situated on the tributary, goes with the stream 80 km to reach the point B, where the tributary flows into the river, and then goes upstream of the river to the point C. It takes the boat 18 hours to cover the path from A to C and 15 hours to cover the way back. Find the distance from A to C if it is known that the rate of flow of the river is 3 km/h and the proper speed of the boat is 18 km/h.

Solution. Let x denote the rate of flow of the tributary in kilometres per hour. Then from A to B the motor-boat goes with the speed $(18+x)$ km/h, while from B to A with the speed $(18-x)$ km/h, covering the path from A to B during $\frac{80}{18+x}$ hours, and the path from B to A during $\frac{80}{18-x}$ hours.

Let y be the distance from B to C in kilometres. Moving from B to C, the boat goes with a speed of 15 km/h and from C to B with a speed of 21 km/h, covering the path BC for $\frac{y}{15}$ hours and CB for $\frac{y}{21}$ hours. It takes the boat $\frac{80}{18+x}+\frac{y}{15}$ hours to cover the entire path from A to C, which, by the hypothesis, amounts to 18 hours, and $\frac{80}{18-x}+\frac{y}{21}$ hours to cover the way back, which, by the hypothesis, amounts to 15 hours.

Let us write the system of equations

$$\begin{cases} \dfrac{80}{18+x}+\dfrac{y}{15}=18 \\ \dfrac{80}{18-x}+\dfrac{y}{21}=15, \end{cases}$$

which is readily solved by the substitution method (for instance, it is possible to express y in terms of x using the first equation). We find: $x = 2$, $y = 210$.

Since the distance from A to C is equal to the sum of the distances from A to B (80 km) and from B to C (210 km), the whole path from A to C is equal to 290 km.

Example 7. A truck left the point A for the point B, and an hour later a car left the point A in the same direction. The truck and the car reached the point B simultaneously. If they had left the points A and B simultaneously to meet each other, the meeting would have taken place in an hour and 12 minutes after the start. How much time does it take the truck to cover the path from A to B?

Solution. Let x denote the speed of the truck in kilometres per hour, y the speed of the car, and z the path from A to B in kilometres. Then it takes the truck $\frac{z}{x}$ hours to cover the path from A to B, and it takes the car $\frac{z}{y}$ hours to cover the same path. From the conditions of the problem it follows that $\frac{z}{x}-\frac{z}{y}=1$. Starting from A and B both vehicles move during $\frac{z}{x+y}$ hours before they meet each other that, by the hypothesis, lasts 1 hour and 12 minutes, that is, $\frac{6}{5}$ hours. Thus, we write the system of two equations in three variables:

$$\begin{cases} \dfrac{z}{x}-\dfrac{z}{y}=1 \\ \dfrac{z}{x+y}=\dfrac{6}{5}. \end{cases} \tag{1}$$

Although the number of unknowns exceeds the number of equations, the problem can be solved since we should find not each of the variables x, y, z, but the ratio $\frac{z}{x}$ (the time during which the truck was in motion). Transforming the second equation of the system to $5z = 6x + 6y$, we then write: $5 = 6\frac{x}{z} + 6\frac{y}{z}$.

Setting $u = \frac{z}{x}$, $v = \frac{z}{y}$, we rewrite System (1) as

$$\begin{cases} u - v = 1 \\ \frac{6}{u} + \frac{6}{v} = 5. \end{cases}$$

Solving this system, we get: $u = 3$, $v = 2$. Hence, it takes the truck 3 hours to cover the path from A to B.

Example 8. The path of a cyclist comprises three sections, the length of the first section being 6 times the length of the third section. What is the speed of motion of the cyclist averaged over the entire

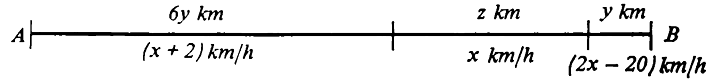

Fig. 1

path if it is equal to his speed along the second section, is 2 km/h less than the speed of his driving along the first section, and is 10 km/h greater than half the speed on the third section?

Solution. Let x denote the average speed of the cyclist in kilometres per hour, y is the length of the third section, and z the length of the second section, in kilometres. Then the speed of the cyclist on the first section is equal to $(x + 2)$ km/h, on the second to x km/h, and on the third to $(2x - 20)$ km/h $\left(\text{since, by hypothesis, the speed } v \text{ of the cyclist on the third section is related with the average speed } x \text{ by the formula: } x = \frac{v}{2} + 10\right)$. Figure 1 represents the scheme of cyclist's motion. The time of his motion from A to B in terms of the introduced variables can be expressed in two ways:

(a) by adding together the time of riding along each of the three sections:

$$\left(\frac{6y}{x+2} + \frac{z}{x} + \frac{y}{2x-20}\right) \text{ hours;}$$

(b) by dividing the whole path by the average speed of the cyclist: $\frac{7y+z}{x}$ hours.

Thus, we write the equation

$$\frac{6y}{x+2}+\frac{z}{x}+\frac{y}{2x-20}=\frac{7y+z}{x}. \qquad (2)$$

Transforming Equation (2), we have:

$$\frac{6y}{x+2}+\frac{y}{2x-20}=\frac{7y+z}{x}-\frac{z}{x},$$

and further

$$\frac{6y}{x+2}+\frac{y}{2x-20}=\frac{7y}{x}.$$

Dividing both sides of the last equation by y (this does not lead to a loss of solutions since, from the physical point of view, $y \neq 0$), we get:

$$\frac{6}{x+2}+\frac{1}{2x-20}=\frac{7}{x},$$

whence we find: $x_1 = 14$, $x_2 = -20$. The second root does not satisfy the conditions of the problem. Hence, the average speed of the cyclist is equal to 14 km/h.

Remark. Equation (2) contains three variables, but in the process of transformations two of them (y and z) were eliminated. Such variables may be called *auxiliary* (we had not to find their values).

Prior to considering the next example, we should like to note that the problems in which certain work is done (for instance, some amount of parts are finished by grinding, a reservoir is filled by a liquid, and so on) may be conventionally regarded as belonging to the class of problems on motion. In problems of this type, the total amount of work done (the number of parts, capacity of the reservoir, etc.) plays the role of distance, while productivity of labour (that is, the amount of work done per unit time) plays the role of speed.

Example 9. Two pipes of different diameters supply a tank with water. On the first day, both pipes, working simultaneously, fed 14 m^3 of water. On the second day, only the smaller pipe was brought into use. It fed another 14 m^3 of water, but operated 5 hours longer than on the first day. On the third day, the operation of the pipes lasted as long as on the second day, but at first both pipes were brought into use and fed 21 m^3 of water and then only the larger pipe continued operating and fed another 20 m^3 of water. How much water is fed by each pipe per hour?

Solution. Let x denote the capacity of the larger pipe, measured in m^3/h, y the capacity of the smaller pipe, measured in m^3/h, t the operation time in hours of both pipes on the first day. Then, on the first day, both pipes fed $(x + y)\, t$ m^3 of water that, by hypothesis, amounts to 14 m^3. Thus, we get the first equation: $(x + y)\, t = 14$.

On the second day, the smaller pipe operated for $(t + 5)$ hours and supplied $y(t + 5)$ m^3 of water that, by hypothesis, is 14 m^3. Hence, we get the second equation: $y(t + 5) = 14$. On the third day, both pipes fed 21 m^3 of water, hence, their joint operation lasted $\frac{21}{x + y}$ hours. Then only the larger pipe continued operating and fed another 20 m^3 of water, hence, its operation lasted $\frac{20}{x}$ hours. Since on the third day the operation lasted as long as on the second day, we get the third equation: $\frac{21}{x + y} + \frac{20}{x} = t + 5$. Thus, we have the following system of equations:

$$\begin{cases} (x+y)\,t = 14 \\ y\,(t+5) = 14 \\ \dfrac{21}{x+y} + \dfrac{20}{x} = t + 5. \end{cases}$$

From the second equation we find: $t + 5 = \frac{14}{y}$, then the first equation can be rewritten as follows: $\frac{14}{x+y} = \frac{14}{y} - 5$, and the third one in the form: $\frac{21}{x+y} + \frac{20}{x} = \frac{14}{y}$.

Thus, we get the system of two equations:

$$\begin{cases} \dfrac{14}{x+y} = \dfrac{14}{y} - 5 \\ \dfrac{21}{x+y} + \dfrac{20}{x} = \dfrac{14}{y}. \end{cases}$$

Getting rid of the denominators in both equations, we have:

$$\begin{cases} 5xy + 5y^2 = 14x \\ 14x^2 - 27xy - 20y^2 = 0. \end{cases}$$

The second equation of the system is homogeneous. Dividing both of its sides by y^2 termwise and setting $z = \frac{x}{y}$, we get the quadratic equation $14z^2 - 27z - 20 = 0$, whose roots are: $z_1 = \frac{5}{2}$, $z_2 = -\frac{4}{7}$. The second root does not satisfy the conditions of the problem, hence, $z = \frac{5}{2}$, i.e. $\frac{x}{y} = \frac{5}{2}$.

It remains to solve the system $\begin{cases} \dfrac{x}{y} = \dfrac{5}{2} \\ 5xy + 5y^2 = 14x, \end{cases}$ whence $x = 5$,

$y=2$, that is, the capacity of the larger pipe is 5 m^3/h, and that of the smaller is 2 m^3/h.

4. Problems on Joint Operation. The contents of problems of this type is usually reduced to the following. Some work whose amount is not indicated and is not sought for (for instance, typing a manuscript, digging a pit, filling a reservoir, etc.) is done by several persons or mechanisms operating uniformly (that is, with a constant output for each of them). In such problems, the total amount of work to be done is taken as 1 (as a unit of measurement).

If productivity of labour, that is, the amount of work done per unit time, is denoted by v, and the time required for completing the total amount of work by t, then $v = \frac{1}{t}$.

Example 10. It takes the first tractor 2 hours less than the third, and 1 hour more than the second tractor to plough the entire field. If the first and second tractors operate together, then the field can be ploughed for 1 hour and 12 minutes. How much time does it take the three tractors to plough the field if they operate jointly?

Solution. Let x hours denote the time necessary for the first tractor to plough the field, y hours for the second, and z hours for the third one. The amount of work (here, this is the area of the field) is taken as 1. Then $\frac{1}{x}$ is the output of the first tractor, $\frac{1}{y}$ is that of the second, and $\frac{1}{z}$ is that of the third. By the hypothesis, $z - x = 2$ and $x - y = 1$. Besides, it is known that if the first and second tractors operate jointly, the entire field can be ploughed for 1 hour and 12 minutes, that is, in $\frac{6}{5}$ of an hour. But in $\frac{6}{5}$ hours, the first tractor does $\frac{6}{5} \times \frac{1}{x}$ of the entire job and the second one does $\frac{6}{5} \times \frac{1}{y}$. Hence, $\frac{6}{5x} + \frac{6}{5y} = 1$.

In the final analysis, we obtain a system of three equations in three variables:

$$\begin{cases} z-x=2 \\ x-y=1 \\ \frac{6}{5x}+\frac{6}{5y}=1. \end{cases}$$

Solving this system, we get: (3, 2, 5), (—0.4, —0.6, 2.4). Obviously, only the first solution satisfies the conditions of the problem.

Let us now answer the question of the problem. The output of the three tractors operating jointly amounts to $\frac{1}{3} + \frac{1}{2} + \frac{1}{5}$, i.e. $\frac{31}{30}$. Hence, it takes the three tractors $\frac{30}{31}$ of an hour to plough the field.

Example 11. When operating together, the combines possessed by a State farm can complete harvesting for 24 hours. But, according to the schedule, they began working in succession: only one combine operated for the first hour, two combines for the second hour, three for the third, and so on, until all the combines were put in operation to work jointly for several hours to complete harvesting. The schedule operation time could be reduced by 6 hours if all the machines, with the exception of five, harvested continuously from the very beginning. How many combines does the State farm have?

Solution. Let us assume the total amount of work to be 1 and introduce three variables, n denoting the number of combines in the State farm, x the output of a combine per hour, and t the time of joint operation of all the combines according to the schedule (in hours). By the hypothesis, n combines, each having the output equal to x, can complete harvesting during 24 hours, that is, $24nx = 1$.

According to the schedule, only one combine operated for the first hour, the work done during this hour being equal to x. Two combines operating for the second hour did the work $2x$. The work done by three combines for the third hour is equal to $3x$, and so forth. During the $(n-1)$th of an hour $(n-1)$ combines did the work equal to $(n-1)\,x$. Then all n combines took part in harvesting during t hours. The work done by them is equal to ntx. Thus, the schedule operation of the combines is described by the equation:

$$x + 2x + \ldots + (n-1)\,x + ntx = 1. \tag{3}$$

Note that $x + 2x + \ldots + (n-1)\,x$ is the sum of $(n-1)$ terms of the arithmetic progression (a_n), in which $a_1 = x$, $d = x$. Hence,

$$x + 2x + \ldots + (n-1)\,x = \frac{x + (n-1)\,x}{2}\,(n-1) = \frac{n\,(n-1)\,x}{2}$$

and Equation (3) takes the form: $nx\left(\frac{n-1}{2} + t\right) = 1$.

Finally, the hypothesis implies that if $(n-5)$ combines had operated from the very beginning, the harvesting would not have lasted $(n-1+t)$ hours, as stipulated by the schedule, but 6 hours less, that is, $((n-1)+t-6)$ hours. Therefore, $(n+t-7)\,(n-5)\,x = 1$.

As a result, we get the system of three equations in three variables n, x, t:

$$\begin{cases} 24nx = 1 \\ nx\left(\frac{n-1}{2} + t\right) = 1 \\ (n+t-7)\,(nx-5x) = 1. \end{cases}$$

From the first equation we find: $nx = \frac{1}{24}$. Substituting this expression into the second and third equations, we get:

$$\begin{cases} nx = \frac{1}{24} \\ \frac{n-1}{2} + t = 24 \\ (n+t-7)\left(\frac{1}{24} - 5x\right) = 1. \end{cases}$$

Then the system is readily solved by the substitution method. From the first equation we find: $x = \frac{1}{24n}$, from the second: $t = \frac{49-n}{2}$. Substituting these expressions into the third equation, we get:

$$\frac{(n+35)(n-5)}{48n} = 1,$$

whence we find: $n = 25$ (the second solution does not satisfy the cone ditions of the problem). Hence, there were 25 combines in the State farm.

5. Problems on Alloys and Mixtures. Problems of this type are concerned with making up mixtures, alloys, solutions, etc. The solution of such problems is connected with notions such as concentration, percentage, sampling, humidity, and so on, and is based on the following assumptions:

1. All mixtures (alloys, solutions) obtained are homogeneous.
2. No distinction is made between a litre as a unit of capacity and a litre as a unit of mass.

If a mixture (alloy, solution) of mass m consists of substances A, B, C (whose masses are m_1, m_2, m_3, respectively), then the quantity $\frac{m_1}{m}$ $\left(\frac{m_2}{m}, \frac{m_3}{m}, \text{respectively}\right)$ is called the *concentration* of the substance A (B, C, respectively) in the mixture. The quantity $\frac{m_1}{m}100\%$ $\left(\frac{m_2}{m}100\%, \frac{m_3}{m}100\%, \text{respectively}\right)$ is called the *percentage* of the substance A (B, C, respectively) in the mixture. It is clear that $\frac{m_1}{m} + \frac{m_2}{m} + \frac{m_3}{m} = 1$, that is, the concentration of the third substance depends on the concentration of the first two.

Example 12. A 12-kg piece of alloy of copper and tin contains 45% copper. How much pure tin should be added to this alloy to get a new alloy containing 40% copper?

Solution. Let the mass of tin to be added to the original alloy be x kg. Then we get a new alloy, whose weight is $(12 + x)$ kg, containing 40% copper. Hence, the new alloy contains $\frac{12 + x}{100}$ 40 kg

of copper. The original alloy (whose mass was 12 kg) contained 45% copper, that is, its copper content amounted to $\frac{12}{100} 45$ kg. Since the mass of copper remains unchanged in both alloys, we may write the following equation:

$$\frac{(12+x)\,40}{100} = \frac{12}{100} 45.$$

Solving this equation, we get: $x = 1.5$. Thus, 1.5 kg of tin must be added to the original alloy.

Example 13. There are two sorts of steel with nickel contents 5% and 40% by mass. How much steel of each sort must be taken for remelting to get 140 tonnes of steel containing 30% nickel?

Solution. Let the mass of the steel of the first sort be x tonnes. Then we must take $(140 - x)$ tonnes of the steel of the second sort. The steel of the first sort contains 5% nickel, hence, x tonnes of steel contain $x \times 0.05$ tonnes of nickel. The nickel content of the steel of the second sort is 40%, hence, $(140 - x)$ tonnes of this steel contain $(140 - x)\,0.4$ tonnes of nickel. The question states that after remelting the two steel samples, we get 140 tonnes of the steel containing 30% nickel, that is, 140×0.3 tonnes of nickel. But we know that this mass of nickel is the sum of the mass contents of the metal in both sorts of steel, that is, $0.05x$ tonnes and $(140 - x)\,0.4$ tonnes. Thus, we write the equation

$$0.05x + (140 - x)\,0.4 = 140 \times 0.3,$$

whence we find: $x = 40$. Consequently, we must take 40 tonnes of the steel containing 5% nickel and 100 tonnes of the steel with 40% nickel.

Example 14. Several litres of acid was poured from a 54-litre vessel and the same volume of water was added instead, after which the same volume of mixture was poured again. As a result, the mixture in the vessel contained 24 litres of pure acid. How much acid was poured initially?

Solution. Let x litres of acid be poured out initially. Then $(54 - x)$ litres of acid remained in the vessel. Having added water, we obtained 54 litres of the mixture containing $(54 - x)$ litres of acid. Hence, one litre of the mixture contains $\frac{54 - x}{54}$ litres of acid (concentration of the solution). Then x litres of the mixture containing $\frac{54 - x}{54} x$ litres of acid was poured off the vessel. The total amount of acid removed from the vessel is equal to $54 - 24 = 30$ litres. Hence, we get the equation:

$$x + \frac{54 - x}{54} x = 30.$$

Solving this equation, we find two roots: $x_1 = 90$, $x_2 = 18$. It is clear that the value $x_1 = 90$ does not satisfy the conditions of the problem. Consequently, 18 litres of acid was poured initially.

Example 15. An 8-litre vessel is filled with a mixture of oxygen and nitrogen, the oxygen content by volume being 16%. Some of the mixture is released from the vessel, and nitrogen is added instead. Then the same amount of the mixture is released, and nitrogen is added for the second time. As a result, the oxygen content in the vessel became 9%. How many litres of the mixture was released each time?

Solution. Suppose that each time x litres of the mixture was released, and x litres of nitrogen was added. After the first discharge the vessel contained $(8 - x)\,0.16$ litres of oxygen dissolved in 8 litres of the mixture (after adding nitrogen initially). The oxygen concentration at this step was $\frac{(8-x)\,0.16}{8}$, i.e. $(8 - x)\,0.02$. After x litres of the mixture was let out for the second time, $(8 - x)$ litres of the mixture remained in the vessel with the oxygen concentration equal to $(8 - x)\,0.02$, that is, $(8 - x)\,(8 - x)\,0.02$ litres of oxygen was dissolved in 8 litres of mixture. The concentration of oxygen at this step was $\frac{(8-x)^2 0.02}{8}$, its percentage being $\frac{(8-x)^2}{8}\,0.02 \times 100\%$. Hence, we get the equation

$$\frac{(8-x)^2\,0.02}{8}\,100 = 9,$$

whence we find: $x_1 = 2$, $x_2 = 14$. It is clear that it is impossible to release 14 litres from an 8-litre vessel. Hence, 2 litres of the mixture was released from the vessel each time.

Example 16. Two alloy samples with masses a and b kilograms contain different percentage of copper. Two pieces of equal masses were cut from each of the alloy samples and fused together with the remainders of the other samples. The copper content of the two new alloys then turned out to be the same. Find the mass of each of the pieces that were cut.

Solution. Let x be the mass of each of the cut-off pieces in kilograms, y the copper percentage in the first alloy, and z the copper percentage in the second alloy.

After the x-kg pieces were switched around, the first new alloy (see Fig. 2) contained $\frac{a-x}{100}\,y + \frac{x}{100}\,z$ kg of copper, the percentage of copper being equal to $\frac{\frac{a-x}{100}\,y + \frac{x}{100}\,z}{a}\,100\%$, that is, to $\frac{(a-x)\,y + xz}{a}$.

The second new alloy (Fig. 3) contained $\frac{b-x}{100}z+\frac{x}{100}y$ kg of copper, the percentage of copper being equal to $\frac{\frac{b-x}{100}z+\frac{x}{100}y}{b}100\%$, that is, to $\frac{(b-x)z+xy}{b}$. By the hypothesis, the two newly obtained

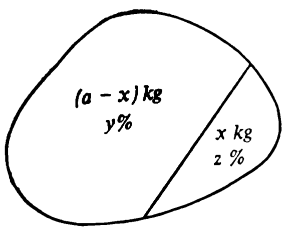

Fig. 2

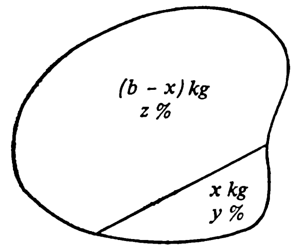

Fig. 3

alloys have an equal copper percentage. Hence, we get the equation

$$\frac{(a-x)y+xz}{a}=\frac{(b-x)z+xy}{b}.$$

We have in succession:

$$aby - bxy + bzx = abz - axz + axy,$$
$$(aby - abz) - (bxy - bxz) - (axy - axz) = 0,$$
$$ab(y - z) - bx(y - z) - ax(y - z) = 0,$$
$$(y - z)(ab - ax - bx) = 0.$$

By the hypothesis, $y \neq z$, hence, $ab - ax - bx = 0$, whence we find: $x = \frac{ab}{a+b}$. The variables y and z were eliminated in the process of solving the obtained equation (auxiliary variables).

Hence, the mass of each of the cut-off pieces is equal to $\frac{ab}{a+b}$ kg.

EXERCISES

480. The sum of the squares of the digits of a two-digit number is equal to 10. Subtracting 18 from the original number, we obtain a number written with the same digits but in the reverse order. Find the original number.
481. What two-digit number is four times the sum of its digits and three times the product of its digits?
482. Find two integers whose sum is equal to 1244. If the digit 3 is annexed to the right of the first number, and the last digit 2 is rejected from the second number, then the newly obtained numbers will be equal to each other.

483. A three-digit number ends in the digit 3. If this digit opens the number, then the newly obtained number will exceed the triple original number by 1. Find the original number.

484. A six-digit number begins with the digit 2. If this digit is moved from the first to the last place without violating the sequence of the remaining digits, then the newly obtained number will be three times the original number. Find the original number.

485. The sum of all even two-digit numbers is divided without a remainder by one of them. Find the divisor if it is known that the sum of its digits is equal to 9, and that the quotient differs from the divisor only by the sequence of the digits.

486. The division of a two-digit number by the sum of its digits yields 7 with 6 as a remainder. If this two-digit number is divided by the product of its digits, then we get 3 as a quotient with a remainder equal to the sum of the digits of the original number. Find the original two-digit number.

487. The sum of two three-digit numbers is equal to 1252; both numbers are formed by the same digits following in reverse order. Find these numbers if the sum of the digits of each number is equal to 14, and the sum of the squared digits is 84.

488. A sportsman climbing a mountain reaches an altitude of 800 m by the expiry of the first hour. Each subsequent hour he ascends to a height by 25 m less than during the preceding hour. How many hours does it take the sportsman to reach an altitude of 5700 m?

489. The division of the ninth term of an arithmetic progression by its second term yields 5, and the division of the thirteenth term of this progression by its sixth term yields 2 as a quotient and 5 as a remainder. Find the sum of the first 20 terms of this progression.

490. The sum of an infinitely decreasing geometric progression is equal to 4, and the sum of its cubed terms is equal to 192. Find the first term and the common ratio of the progression.

491. Find four numbers, the first three of which form an arithmetic progression and the last three numbers form a geometric progression; the sum of the first and fourth numbers is equal to 66, and the sum of the second and third to 60.

492. The sum of the first three terms of a geometric progression is equal to 91. Adding 25, 27, and 1 to these terms, respectively, we get three numbers forming an arithmetic progression. Find the seventh term of the geometric progression.

493. Find a three-digit number whose digits form a geometric progression. Subtracting 792 from this number, we get a number written with the same digits but in reverse order. Subtracting 4 from the hundreds digit of the number to be found and leaving the rest of the digits unchanged, we get a number whose digits form an arithmetic progression.

494. Find the four-digit number, the first three digits of which form an increasing arithmetic progression if it is known that it is divisible by 225.

495. Three brothers share some money in proportion to their age. The numbers expressing their age form a geometric progression. If they shared this money in proportion to their age in three years, then the youngest would get 105 roubles more and the middle brother 15 roubles more than now. How old is each brother if it is known that the difference in age between the oldest and youngest is equal to 15 years?

496. Find the number of terms of the arithmetic progression if the ratio of the sum of the first 13 terms to the sum of the last 13 terms is equal to $\frac{1}{2}$, and the ratio of the sum of all the terms, less the first three, to the sum of all the terms, less the last three, is equal to $\frac{4}{3}$.

497. The sum of all the terms of a decreasing geometric progression is equal to $\frac{16}{3}$. The progression contains a term equal to $\frac{1}{6}$. The ratio of the sum of all the terms preceding this term to the sum of those following this term is equal to 30. Determine the number of the term equal to $\frac{1}{6}$.

498. The mass of an alloy is 2 kg. It consists of silver and copper, the mass of silver amounting to $14\frac{2}{7}$ % of the mass of copper. How much silver is there in the alloy?

499. One metre each of two different kinds of cloth cost a total of 15 roubles and 20 kopecks. If the price of the cloth of the first kind were higher and of the second kind lower than the real price by the same percentage, then one metre of the cloth of the first kind would have cost 15 roubles and of the second—2 roubles and 40 kopecks. What is the price of one metre of cloth of the first kind?

500. A one-digit number was increased by 10. If now the obtained number is increased by the same percentage as in the first increase, then the result will be 72. Find the original one-digit number.

501. According to their plan, two plants had to turn out 360 machines during a month. The first plant fulfilled its plan by 112%, and the second by 110%, having produced collectively 400 machines during this month. How many machines were manufactured in excess of the plan by each plant?

502. To bake wheat-bread, a baker took a certain amount of flour equal (in kg) to the percentage of the gain in weight of bread for this amount of flour. To bake rye-bread, he took 10 kg more flour than for wheat-bread so that the mass of the flour (in kilograms) was equal to the gain for rye-flour. How much wheat- and rye-flour was taken if the total amount of baked bread was 112.5 kg?

503. The working day of eight hours is reduced by an hour. How many percent should productivity of labour be increased so that the overall wages increase by 5% without changing the piece-rates?

504. At the beginning of a year, 1600 roubles was deposited in a savings-bank, and at the end of this year 848 roubles was withdrawn. At the close of the second year, 824 roubles turned out to be in the savings account. What is the interest rate set by the savings-bank per annum?

505. At the end of a year a savings-bank calculated the interest due to the depositor as 6 roubles. Adding another 44 roubles, the depositor left his money for another year. At the end of the second year the due interest was calculated once again, and now the deposit together with all the due interest amounted to 257 roubles and 50 kopecks. How much money was deposited originally?

506. The price of an item was reduced by 20%, then the new price was reduced by 15%; finally, after taking a fresh inventory of the goods, the price was cut by another 10%. By how many percent was the original price reduced as a result of the three cuts?

507. The number of students at an institute, increasing by the same percentage each year, grew from 5000 to 6655 over a three-year period. By how many percent did the number of students increase yearly?

508. The volume of substance A is half the sum of the volumes of substances B and C, and the volume of substance B is 20% of the sum of the volumes of substances A and C. Find the ratio of the volume of substance C to the sum of the volumes of substances A and B.

509. As a result of reconstruction of a factory, the number of the disengaged workers was within the limits from 1.7% to 2.3% of the total number of the personnel. Find the minimal number of workers which could be employed before the reconstruction.

510. The number of the students in the group having passed the exams is within the limits from 96.8 to 97.2% of the total number of the students. Find the minimal number of the students in such a group.

511. A man had to cover the distance from a village to a railway station. Having covered 3 km during the first hour, he understood that he could be late for the train and began walking at a speed of 4 km/h. He reached the station 45 minutes before the train departed. If he had gone at a speed of 3 km/h, he would have been 40 minutes late for the train. Determine the distance from the village to the station.

512. A passenger in a train which moves with a speed of 40 km/h noted that another train passed his window in the opposite direction for 3 seconds. What is the speed of the second train if its length is 75 metres?

513. A cyclist had to ride 48 km with a certain average speed. But for some reason the first half of his way he rode with a speed reduced by 20%, and the second half with a speed exceeding the supposed average speed by 2 km/h. It took the cyclist 5 hours to cover the entire distance. Find the supposed average speed.

514. Three bodies move in the same straight line from point A to point B. The second body began moving 5 seconds, and the third body 8 seconds later than the first one. The speed of the first body is less than that of the second body by 6 cm/s. The speed of the third body is equal to 30 cm/s. Find the distance AB and the speed of the first body if it is known that all the three bodies reach the point B at the same instant of time.

515. A plane first flew with a speed of 220 km/h. When it still had to fly 385 km less than the covered distance, its speed became equal to 330 km/h. The average speed of the plane during the flight was equal to 250 km/h. What distance was covered by the plane?

516. Two trains left points A and B simultaneously to meet each other. The speed of the first train exceeded the speed of the second by 10 km/h. The trains met at a point 28 km away from the midpoint of AB. If the first train had left A 45 minutes later than the second, then the trains would have met at the midpoint of AB. Find the distance AB and the speeds of both trains.

517. Two schoolboys left their house at the same time and walked to school at the same speed. Three minutes later one of them remembered that he had forgotten a necessary book and ran home at a speed exceeding the initial speed by 60 m/min. He took the book and ran to school at the same speed. He caught up with his friend, who was walking at a constant speed, at the school's entrance. Find the speeds of the schoolboys if the distance from the school to their house is equal to 280 m.

518. Two pedestrians start simultaneously from points A and B which are 27 km apart and move along straight line AB. If they move in opposite directions, they meet in 3 hours, while walking in the same direction, one catches up with the other in 9 hours. Find the speed of each pedestrian.

519. Two bodies move along two sides of a right angle towards its vertex. At the initial instant the body A was 60 m away from the vertex, while the body B—80 m. In 3 seconds, the distance between A and B became equal to 70 m, and in another 2 seconds to 50 m. Find the velocity of each body.

520. The distance between two towns situated on the bank of a river is equal to 80 km. It takes a motor-boat 8 hours and 20 minutes to cover this distance twice (upstream and downstream). Determine the speed of the motor-boat in stagnant water if the rate of flow of the river is 4 km/h.

521. A motor-boat went 8 km against the stream, then turned and went 36 km with the stream. The whole trip lasted for 2 hours. Then the motor-boat went 6 km against and 33 km with the stream. This second trip lasted for 1 hour and 45 minutes. Find the speed of the launch in stagnant water.

522. Two rivers flow into a lake. A motor-boat leaves the landing-stage A situated on the first river, goes 24 km downstream to reach the lake, sails along the lake for two hours, and then goes 32 km along the second river and reaches the landing-stage B. It took the motor-boat 8 hours to cover the whole path from A to B. If the motor-boat had sailed 18 km more along the lake, it would have covered the whole path in 10 hours. Find the rate of flow of each river if the rate of flow of the first river is known to be 2 km/h greater than the rate of flow of the second river.

523. Two pedestrians started simultaneously from points A and B to meet each other. When the first pedestrian covered half the path, there remained 24 km for the second pedestrian to complete his walk. When the second pedestrian covered half the path, the first was at a distance of 15 km from the finish. How many kilometres will it remain for the second pedestrian to reach A after the first completes the path from A to B?

524. Two trains left from points A and B to meet each other; the second train departed half an hour later than the first. In two hours after the departure of the first train the distance between them was equal to $\frac{19}{30}$ the distance between A and B. The trains met at the midpoint of the path AB. How much time will it take each train to cover the distance AB?

525. The distance between two towns A and B is equal to 60 km. Two trains start simultaneously: one from A to B, the other from B to A. Having covered 20 km, the train moving from A to B stopped for half an hour and then, continuing its movement for 4 minutes, met the train coming from B. Both trains reached their destination simultaneously. Find the speed of each train.

526. Two cyclists started simultaneously from points A and B to meet each other. The one driving from A reached B in four hours, and the other driving from B reached A in nine hours after they had met. How much time does it take each cyclist to cover the distance?

527. A motor-boat left point A to go against the stream of a river, and a raft started simultaneously from point B situated upstream from point A. In a hours they met and continued moving without stops. Having reached B the motor-boat, without any delay, turned, began its return trip, and caught up with the raft at point A. How much time does it take the raft and the motor-boat to meet at point A if the proper speed of the motor-boat is known to be constant?

528. A fast train covers the distance between two towns 4 hours faster than a goods train and 1 hour faster than a passenger train. It is known that the speed of the goods train is $\frac{5}{8}$ of the speed of the passenger train and 50 km/h less than the speed of the fast train. Find the speeds of the goods and fast trains.

529. A passenger train and a fast train left simultaneously two points which are 2400 km apart to meet each other. Each of them moves with a constant speed, and at a certain instant they meet. If both trains had moved with the speed of the fast train, then they would have met three hours earlier. If both trains had moved with the speed of the passenger train, then their meeting would have taken place five hours later than it actually did. Find the speeds of the trains.

530. Two points move in a circle whose circumference is equal to 360 m, the first point completing the circle 1 second earlier than the second point. Find the velocity of either point if it is known that during one second the first point covers 4 m more than the second point.

531. When moving in a circle in the same direction, two points meet each other every 20 seconds, and when moving in opposite directions, they meet

every 4 seconds. Find the velocity of either point if it is known that the circumference of the circle is equal to 100 m.

532. When moving in a circle in the same direction, two points meet each other every 56 minutes, and when moving in opposite directions—every 8 minutes. Find the velocity of each point and the circumference of the circle if it is known that during one second the first point covers $\frac{1}{12}$ m more than the second point.

533. When moving in a circle in the same direction, two points meet every 12 minutes, the first point completing the circle 10 seconds faster than the second point. What part of the circumference does each point cover per second?

534. A ship left port A for port B, and 7.5 hours later a motor-boat followed the ship in the same direction. Halfway between A and B the boat caught up with the ship. When the boat reached B, the ship had still to cover $\frac{3}{10}$ of the entire distance. How much time will it have taken the ship to cover the distance from A to B?

535. A stopping train left point A for point B; three hours later an express followed the stopping train. The express overtook the stopping train halfway between A and B. By the time the express reached B, the stopping train had covered $\frac{13}{16}$ of the total route. How much time will it have taken the stopping train to travel from A to B?

536. A pedestrian left A for B and $\frac{3}{4}$ hour later a cyclist followed him. When the cyclist reached point B, the pedestrian had $\frac{3}{8}$ of the entire path to walk. How much time would it take the pedestrian to walk from A to B if the cyclist caught up with the pedestrian halfway between A and B?

537. A cyclist left point A for point B, which are 70 km apart; some time later a motor-cyclist followed him having also started from point A and travelled at 50 km/h. The motor-cyclist caught up with the cyclist 20 km away from point A. He reached point B and 48 minutes later turned back toward A. He again came across the cyclist, who had by then been travelling from A to B for 2 hours and 40 minutes. Find the speed of the cyclist.

538. A boat and a raft started simultaneously moving downstream from a landing-stage A on the bank of a river. The boat reached another landing-stage B, 324 km away from A, and after 18 hours left B to return to A. When the boat was 180 km away from landing-stage A a second boat having left A 40 hours later than the first one overtook the raft which had by then covered 144 km. Find the speeds of each boat if it is known that they are equal, and the speed of the current.

539. A tributary flows into a river. A boat leaves a landing-stage situated on the tributary and moves 60 km downstream to reach the junction. It then moves 65 km downstream along the river to reach another landing-stage B. Then following the same route the boat returns to landing-stage A. It takes the boat 10 hours to return. Find the proper speed of the boat if it is known that it takes the boat 3 hours and 45 minutes to get from A to the river, and that the flow rate of the river is 1 km/h less than that of the tributary.

540. Two swimmers started one after the other in a 50-metre pool to cover a distance of 100 m. The speed of the second swimmer was 1.5 m/s. Having covered 21 m he caught up with the first swimmer, reached the opposite wall of the pool, returned back, and met the first swimmer $\frac{2}{3}$ second after the turn. Find the time interval between their starts.

541. Two skiers started from point A in the same direction, the second skier starting six minutes after the first and overtook the first skier 2 km from the start. Having covered 5 km in all, the second skier returned back and met the first skier 4 km from the start. Find the speed of the second skier.

542. Two cyclists started, the second 2 minutes after the first. The second cyclist overtook the first 1 km from the start. If the second cyclist, having covered another 5 km, returned back, then he would meet the first cyclist 20 minutes after the first cyclist had started cycling. Find the speed of the second cyclist.

543. A cyclist and a pedestrian simultaneously leave A for B. The cyclist can go twice as fast as the pedestrian. At the same time, another pedestrian leaves B for A to meet them. The time interval between the moment he meets the cyclist and the moment he meets the first pedestrian comprises $\frac{2}{15}$ of the time it takes him to walk from B to A. Which of the pedestrians walks the faster and by how many times, given that both of them had walked more than $\frac{1}{4}$ of the distance from A to B before they met?

544. A ship left point A for point B. At 8 o'clock the ship overtook a boat moving at 3 km/h in the same direction. Having stayed for 10 minutes at B, the ship returned to A, meeting the boat at 8:20. The ship reached A at the same time the boat reached B. Determine the time the boat arrived at point B if it had been 1.5 km from point A at 8:10.

545. If a passenger goes from point A by train, he will reach point B in 20 hours. If he goes by plane, he will have to wait for two hours, but he will reach B in 10 hours after the train has departed. How many times faster is the plane than the train if $\frac{8}{9}$ of an hour after the plane has taken off both of them are at the same distance from A?

546. A pedestrian and a cyclist simultaneously left point A for point B. Having reached B, the cyclist turned and an hour after starting met the pedestrian. The pedestrian continued walking toward B, while the cyclist turned once more and also rode toward B. Upon reaching B, the cyclist turned and rode back to A to meet the pedestrian 40 minutes after their first meeting. How much time does it take the pedestrian to walk from A to B?

547. Three cyclists started from point A. The first cyclist left an hour earlier than the other two, who started together. Some time later the third cyclist caught up with the first, while the second cyclist overtook the first two hours after the third cyclist had done so. Determine the ratio between the speeds of the first and third cyclists if the ratio between the speeds of the second and third cyclists is 2:3.

548. Two points A and B are 105 km apart. A bus left A for B at a speed v km/h. Thirty minutes later a car travelling at 40 km/h followed the bus. Having overtaken the bus, the car returns to A at the same speed. For what range of values of v will the bus reach B before the car arrives at A?

549. Two messengers left points A and B simultaneously to meet each other. After some time they met. Had the first messenger started an hour earlier, and the second messenger half an hour later, then they would have met 48 minutes earlier. Had the first messenger started half an hour later, and the second messenger an hour earlier, then the place where they met would have been 5600 m closer to A. Find the speed of each messenger.

550. Point C lies between points A and B, viz. $AC = 17$ km, $BC = 3$ km. A car left A for B. Having covered less than two kilometres, it stopped for some time. When it started moving again toward B, a pedestrian and a cyclist left C for B and, after having reached B, turned toward A. Who will meet the car first if the car is four times faster than the cyclist and eight times faster than the pedestrian?

551. A pedestrian left point A for point B. At the same time a motor-cyclist started from B toward A to meet the pedestrian. After meeting the pedestrian the motor-cyclist took the pedestrian to B and returned to A at once. As a result, the pedestrian got to B four times faster than he had planned. How many times faster would the motor-cyclist have reached point A if he had not had to return?

552. A load was delivered from point A to point B. First it was transported by van and then by truck. The distance between where the load was transferred and point B is one-third of the distance between it and point A. The time it took for the load to be taken from A to B is the same as the time it could have taken had the load been taken directly from A to B at 64 km/h. How fast did the truck travel if the speed of the van is known not to have exceeded 75 km/h? In addition, if the van and the truck had left A and B to meet each other, then they would have met after a time interval that would have elapsed had the load been taken directly from A to B at 120 km/h.

553. Two cyclists started simultaneously from points A and B and met each other 2.4 hours later. Had the first cyclist travelled 50% faster and the second 20% faster, then it would have taken the first cyclist $\frac{2}{3}$ hour more than the second cyclist to ride from A to B. How much time does it take each cyclist to ride from A to B?

554. A motor-cyclist left point A for point B. Two hours later a car followed him and reached B at the same time as the motor-cyclist. Had the car and the motor-cyclist started from A and B simultaneously to meet each other, then they would have met 1 hour and 20 minutes after they started. How much time does it take the motor-cyclist to travel from A to B?

555. A cyclist left A for B. Simultaneously, a motor-scooter started from B and met the cyclist after 45 minutes. How much time does it take the cyclist to ride from A to B if the motor-scooter can travel the same distance 2 hours faster?

556. It takes a ship three hours to go from A to B and 4 hours to return. How long would it take a raft to float from A to B?

557. A maintenance man takes 30 seconds to run down a moving escalator. It takes him 45 seconds to descend along the escalator when motionless. How long would it take him to descend by simply standing on the escalator when moving?

558. A lorry left point A for point B. An hour later it was followed by a car which also started from A. Both vehicles reached point B simultaneously. Had they started simultaneously from A and B to meet each other, they would have met 1 hour and 12 minutes after the start. How much time does it take the lorry to ride from A to B?

559. A cyclist and a bus simultaneously left points A and B to meet each other. It takes the cyclist 2 hours and 40 minutes more to go from A to B than the bus to go from B to A, and the sum of the times they take is $\frac{16}{3}$ times the time it takes for them to meet after starting. How much time does it take the cyclist to go from A to B and the bus to go from B to A?

560. Some mail was delivered from point A to point B. First it was carried by a motor-cyclist who, having covered $\frac{2}{3}$ of the distance from A to B, handed it over to a cyclist who was waiting for him. The mail was at B as if it had been taken at an average speed of 40 km/h. Had the motor-cyclist and cyclist left A and B simultaneously to meet each other, they would have met after an interval of time that would have been required to move from A to B at 100 km/h. Find the speed of the motor-cyclist supposing that he is faster than the cyclist.

561. Two sections of a coal mine were in operation when a third was opened. The third section raised the output of the mine by 1.5 times. If the first and third sections together produce for four months as much coal as the second section does in a year, what is the percentage output of the second section in terms of the first section's produce?

562. Two teams began working at 8 a.m. Having made 72 parts together, they continued working separately. At 3 p.m. it turned out that whilst the teams worked separately the first team manufactured 8 parts more than the second team. Next day the first team manufactured one part more per hour, and the second team one part less per hour than on the first day. They began working together at 8 a.m. and after 72 parts had been ready, continued working separately as they had done the day before. This time, the first team manufactured 8 parts more than the second team already by 1 p.m. How many parts were manufactured by each team per hour?

563. A pool is filled with water through one pipe 5 hours faster than it is when the water is passed through a second pipe, and 30 hours faster than when through a third pipe. The capacity of the third pipe is $\frac{1}{2.5}$ of the capacity of the first pipe and 24 m^3/h less than the capacity of the second pipe. Find the capacities of the first and third pipes.

564. Three workers have to make 80 identical parts. Together they manufacture 20 parts per hour. Initially, the first worker got to work and made 20 parts in over three hours. The remaining parts were made by the second and third workers. It took them 8 hours to complete this job together. How much time would it have taken the first worker to make all 80 parts?

565. A tanker was filled with oil through two pipes, each of which having filled more than $\frac{1}{4}$ of its volume. If the amount of oil supplied per hour through the first pipe were increased by 1.5 times, and the amount of oil supplied per hour through the second pipe $\frac{1}{4}$ of its actual capacity, then the time required to fill up the tanker would have been $\frac{1}{6}$ of the time necessary to fill up the tanker through the first pipe only. Which pipe supplies more oil and by how many times?

566. Oil is pumped into a tank through three pipes and pumped out through a fourth pipe. On the first day, the third and fourth pipes operated for six hours each, the second pipe for five hours, and the first pipe for two hours. As a result, the oil level rose 4 m. On the second day, the first and second pipes operated for three hours each, the third for nine hours, and the fourth during four hours. As a result the oil level rose a further 6 m. On the third day, the second and fourth pipes operated for six hours each. Did the oil level rise or fall on the third day?

567. Two workers, operating together, carried out a job in 12 hours. Had the first worker done half the job, and then the second worker the remaining half, the whole job would have been carried out in 25 hours. How much time would it have taken each worker to do the job separately?

568. Two workers fulfil some job. After 45 minutes' joint operation the first worker was given another job, and the second worker completed the remaining part of the job 2 hours and 15 minutes. How much time would it have taken each worker to complete the whole job alone if the first worker had been able to do this an hour earlier than the second worker?

569. Two turners had to manufacture a number of parts. After 3 hours of working together, the second turner, only, continued working for another 4 hours. As a result, he did 12.5% more than was assigned. How much time would it have taken each turner to complete the initial assignment

alone if the second turner could complete it 4 hours earlier than the first turner?

570. A pool is filled with water from two taps. The pool can be filled if the first tap is opened for 10 minutes and the second for 20 minutes. If the first tap is opened for 5 minutes, and the second for 15 minutes, the pool will be $\frac{3}{5}$ filled. How much time would it have taken to fill the whole pool by using each tap singly?

571. Two teams worked together for 15 days. A third team then joined them as a result of which in 5 days the whole job was completed. The daily output of the second team is 20% higher than that of the first team. The second and third teams together could have fulfilled the whole job in $\frac{9}{10}$ of the time it would have taken the first and third teams to do so together. How much time would it have taken the three teams to do the job had they worked together from the start?

572. Two teams of stevedores were to unload a barge. The sum of the times it would take each team working individually to unload the barge is 12 hours. How much time would it take each team to unload the barge if the difference between these times is 45% of the time it would take both teams working together to unload the barge?

573. Two excavators were used to dig a trench. The first excavator needs three hours less to dig the whole trench than the second excavator needs. How many hours does it take each excavator working separately to dig the trench if the sum of the two times is $\frac{144}{35}$ of the time (in hours) it would take the two excavators, operating jointly, to dig the trench?

574. A ship is being loaded by cranes. Four similar cranes worked for the first two hours, then another two smaller cranes joined them, and in three hours the loading was over. If all six cranes had begun operating simultaneously, the loading would have been completed in 4.5 hours. How much time would it take one of the more powerful cranes and one of the less powerful cranes working together to load the ship?

575. Water gradually enters a pit. Ten pumps of equal capacity operating together can pump the water out of the full pit in 12 hours, while 15 such pumps would need six hours. How much time would it take 25 such pumps?

576. Two factories have to process some raw material. If the output of the second factory were doubled, the time needed for the two factories to fulfil the assignment would be $\frac{2}{15}$ of the time needed for the first factory to complete the work alone. Which factory has more output and how many times more if each factory processed at least $\frac{1}{3}$ of the total input?

577. Two teams of workers together dug a trench in two days. Then they began digging another trench of the same depth and width, but five times longer than the first. The first team began digging the trench alone and was then relieved by the second team. The first team dug 1.5 times more than the second team. The second trench was dug in 21 days. How many days would it have taken the second team to dig the first trench if the first team can dig faster than the second team?

578. A tank is filled with water through five pipes. Water flowing through the first pipe fills the tank in 40 minutes, through the second, third, and fourth pipes together in 10 minutes, through the second, third, and fifth pipes together in 20 minutes, and through the fourth, and fifth pipes jointly in 30 minutes. How much time would it take to fill the tank using all five pipes?

579. Three automatic assembly lines turn out the same product, but they each have different outputs. The output of all the three assembly lines, operating simultaneously, is 1.5 times the output of the first and second lines operating jointly. A task assigned to the first line can be carried out by the second and third lines operating simultaneously 4 hours and 48 minutes faster than it can be done by the first line. The same task is fulfilled by the second line 2 hours faster than by the first line. How much time does it take the first line to do the task?

580. Two tractors plough a field separated into two equal areas. Both tractors began working simultaneously, each ploughing its half. Five hours later they had ploughed half of the whole field, leaving $\frac{1}{10}$ of the area for the first tractor to complete, and $\frac{2}{5}$ for the second tractor to finish. How much time would it take the second tractor to plough the whole field?

581. Three excavators are busy digging a pit. The difference between the outputs of the first and third excavators is three times the difference between that of the third and second excavators. The first excavator does $\frac{4}{5}$ of the whole job within a period of time that would be needed for the second excavator alone to fulfil $\frac{1}{15}$ of the whole job and the third excavator alone to do $\frac{9}{28}$ of the remaining work. How much faster than the second does the first excavator work?

582. The same work can be done by three teams. The first team can do $\frac{2}{3}$ of the work in the time it takes the third team alone to do $\frac{1}{3}$ of the work and the second team to do $\frac{9}{10}$ of the rest. The third team can do half as much as the first and second teams working together. How many times greater is the output of the second team over that of the third?

583. Two teams of plasterers, working jointly, plastered a house in six days. Then they plastered a club, doing three times as much work as when they plastered the house. One team began plastering the club and was then relieved by the second team, which completed the job, the first team doing twice as much work as the second. It took both teams 35 days to plaster the club. How many days would it have taken the first team to plaster the house if the second team could have done it in more than 14 days?

584. Someone purchased three items: A, B, and C. If A had been five times cheaper, B two times cheaper, and C 2.5 times cheaper, then the purchase would have cost eight roubles. If A had been two times cheaper, B four times cheaper, and C three times cheaper, then the purchase would have cost 12 roubles. What did the purchase actually cost and which, A or B, is more expensive?

585. When mixing a 40% solution of acid with a 10% solution of acid, 800 g of a 21.25% solution was obtained. How many grams of each solution were mixed?

586. We have 735 g of a 16% solution of iodine in alcohol. We need a 10% solution of iodine. How much alcohol must be added to the solution?

587. There are two sorts of steel, one of which contains 5% nickel by mass and the other 10%. How much steel (in tons) of each sort is needed to obtain an alloy containing 8% nickel if the second steel contains 4 tons more nickel than the first?

588. 500 kg of ore contained a certain amount of iron. After removing 200 kg of slag which contains on average 12.5% by mass of iron, the percentage of iron in the remaining ore increased by 20%. How much iron in mass percent remained in the ore?

589. An ore contains 40% mass impurity, while the metal smelted from it contains 4% impurity. How much metal will 24 tons of the ore yield?

590. When smelted, 40 tons of ore yield 20 tons of metal containing 6% mass impurity. What is the percentage of impurity in the ore?

591. As a result of processing, 38 tons of a second grade raw material containing 25% mass impurity yields 30 tons of the first grade material. What is the percentage of impurity in the first grade material?

592. Fresh mushrooms contain 90% water, while dried mushrooms contain 12%. What mass of dried mushrooms will be obtained from 88 kg of fresh mushrooms?

593. When processing flower nectar into honey, bees extract a considerable amount of water. How much flower nectar must be processed to yield 1 kg of honey if nectar contains 70% water, and the honey obtained from this nectar contains 17% water?

594. Two alloys each contain two metals. The ratio of the metals contained in the first alloy is 1:2, and in the second 3:2. In what ratio must these alloys be taken to obtain a new alloy with a ratio of the metals of 8:7?

595. We have four litres of acid in one concentration and six litres of acid with a different concentration. If all the acid is mixed together, a 35% solution of acid is obtained, and if equal volumes of these solutions are taken, then a 36% solution of acid is obtained. How much acid (in litres) is contained in each of the original solutions?

596. 40 kg of a salt solution is poured into two vessels so that the second vessel contains 2 kg more pure salt than the first vessel. If 1 kg of salt is added to the second vessel, then it will contain twice the amount of salt than is in the first vessel. Find the mass of the solution in the first vessel.

597. There are three ingots. The mass of the first is 5 kg and that of the second 3 kg, and both contain 30% by mass of copper. If the first ingot is smelted with the third, a new ingot containing 56% of copper will be obtained, and if the second ingot is smelted with the third, then a new ingot containing 60% of copper will be obtained. Find the mass of the third ingot and the percentage of copper in it.

598. There are two ingots of a gold and silver alloy. The percentage of gold in the first ingot is 2.5 times that in the second ingot. If both ingots are smelted together, then a new one containing 40% by mass of gold is obtained. How many times more massive is the first ingot than the second if when equal masses of the first and second ingots are smelted together, a new alloy containing 35% by mass of gold is obtained?

599. An alloy of copper and silver contains 2 kg more copper than silver. If a further $\frac{9}{16}$ of the silver in the alloy is added, then the percentage of silver in the new alloy will be equal to that of the copper in the original alloy. Find the mass of the original alloy.

600. One liquid has a temperature $a°$, the other $b°$. Mixing certain amounts of the two liquids, we get a mixture of temperature $c°$. What will be the temperature of a new mixture if the taken amounts of liquid are interchanged?

601. A 12-litre vessel was filled with acid. Some of the acid was poured from this vessel into another of the same capacity, and the second vessel filled with water. After this the first vessel was topped up with the solution from the second vessel. Then 4 litres of the solution was poured from the first vessel into the second vessel, as a result of which the solutions in both vessels turn out to contain equal amounts of pure acid. How much acid was originally poured from the first vessel into the second?

602. Six litres of 64% alcohol was poured into a vessel containing water. After it had been thoroughly stirred, 6 litres of the resultant solution was removed. This operation was repeated three times. How much water did the vessel contain originally if the final alcohol concentration in it was 37%?

603. Six kilogrammes of alloy contains a certain percentage of copper. Eight kilogrammes of another alloy contains one-half the copper in percentage than in the first alloy. A fragment of the first alloy, and a fragment twice the mass of the second were broken off. The fragments were each smelted with the rest of the other alloy. As a result, two new alloys were obtained, which each had the same percentage of copper. Determine the mass of each fragment separated from the two initial alloy bars.

604. Two litres of glycerin was poured from a full vessel, the vessel topped with two litres of water. After stirring, two litres of the mixture was poured from the vessel and another two litres of water was added instead. The mixture thus obtained was stirred up and again, two litres of the mixture was replaced with two litres of water. As a result of these operations, the volume of water in the vessel exceeded the volume of the remaining glycerin by three litres. How many litres of glycerin and water was left in the vessel at the end?

605. We have two tanks, one filled with pure glycerin, and the other with water. Using two three-litre scoops, one for ladling glycerin from the first tank, the other for ladling water from the second tank, glycerin was transferred from the first tank to the second tank, and a scoop-full of the contents of the second was transferred to the first tank. The mixtures were stirred in both tanks and the operation was repeated. As a result, half the volume of the first tank was pure glycerin. Find the capacities of the tanks if their total capacity is 10 times the capacity of the first tank.

606. By fusing together two ingots of pig iron of equal mass and different chromium contents, a new alloy was obtained containing 12 kg of chromium. If the mass of the first ingot had been doubled, the alloy would have contained 16 kg of chromium. The chromium content of the second ingot exceeded that of the first by 5%. Find the percentage of chromium in each ingot of pig iron.

607. There are three alloys, one containing 60% aluminium, 15% copper, and 25% magnesium, the second 30% copper and 70% magnesium, and the third 45% aluminium and 55% magnesium. The alloys must be combined to prepare a new alloy containing 20% copper. What are the minimal and maximal percentages of aluminium that the new alloy might have?

608. Three alloys contain respectively 45% tin and 55% lead; 10% bismuth, 40% tin, and 50% lead; and 30% bismuth and 70% lead. The alloys are to be combined to obtain a new alloy containing 15% bismuth. What are the maximal and minimal percentages of lead that the new alloy might have?

SEC. 12. IRRATIONAL EQUATIONS

Equations containing the variables under the radical sign or raised to a fractional power are said to be *irrational*. Such equations are considered over the field of real numbers. When solving irrational equations we use the following two basic methods: (1) raising both sides of an equation to the same power; (2) introducing new (auxiliary) variables. Sometimes we have to apply some artificial methods. When raising both sides of an equation to the same power, the reader should bear in mind that for an odd n the equations $f(x) = g(x)$ and $(f(x))^n = (g(x))^n$ are equivalent, while for an even n, the latter

is a consequence of the former, that is, when passing from the equation $f(x) = g(x)$ to the equation $(f(x))^n = (g(x))^n$ we can have extraneous roots. For instance, the equation $x - 1 = 3$ has one root $x = 4$, whereas the equation $(x - 1)^2 = 3^2$ has two roots: $x_1 = 4$, $x_2 = -2$, one of which (namely, $x = -2$) is extraneous for the equation $x - 1 = 3$.

When solving irrational equations, we frequently use the formula $(\sqrt[n]{f(x)})^n = f(x)$. In the case of an even n, its application may lead to extending the domain of definition of the given equation (for $(\sqrt[n]{f(x)})^n$ the constraint $f(x) \geqslant 0$ is naturally used for an even n, whereas with $(\sqrt[n]{f(x)})^n$ replaced by $f(x)$ this constraint is removed).

For this (and some other) reasons, when solving irrational equations, we must check found solutions in most cases. Depending on the kind of found solutions (prime or composite) and also on the method of solving an equation, one or another checking technique may be chosen.

1. Solving Irrational Equations by Raising Both Sides of an Equation to the Same Power.

Example 1. Solve the equation

$$\sqrt{x-1} + \sqrt{2x+6} = 6. \tag{1}$$

Solution. Squaring both sides of the equation, we get:

$$x - 1 + 2\sqrt{(x-1)(2x+6)} + 2x + 6 = 36,$$

and further $2\sqrt{2x^2 + 4x - 6} = -3x + 31$.

Squaring the last equation, we get:

$$8x^2 + 16x - 24 = 9x^2 - 186x + 961,$$

and further $x^2 - 202x + 985 = 0$, whence we find: $x_1 = 5$, $x_2 = 197$.

Check. The found roots are readily checked directly by substituting them into Equation (1).

(1) $\sqrt{x_1 - 1} + \sqrt{2x_1 + 6} = \sqrt{5-1} + \sqrt{2 \times 5 + 6} = 6.$

Thus, $x_1 = 5$ is a root of the given equation.

(2) $\sqrt{x_2 - 1} + \sqrt{2x_2 + 6} = \sqrt{197 - 1} + \sqrt{2 \times 197 + 6} \neq 6,$

that is, $x_2 = 197$ is an extraneous root. Thus, $x = 5$ is the only root of the given equation.

Example 2. Solve the equation

$$\sqrt{x^2 + x - 5} + \sqrt{x^2 + 8x - 4} = 5. \tag{2}$$

Solution. Transforming Equation (2) to the form

$$\sqrt{x^2 + x - 5} = 5 - \sqrt{x^2 + 8x - 4}$$

and squaring both sides of the obtained equation, we get:

$$x^2 + x - 5 = 25 - 10\sqrt{x^2 + 8x - 4} + x^2 + 8x - 4.$$

We then single out the radical and collect like terms:

$$10\sqrt{x^2 + 8x - 4} = 7x + 26, \tag{3}$$

squaring both sides of Equation (3), we get:

$$100(x^2 + 8x - 4) = (7x + 26)^2 \text{ or } 51x^2 + 436x - 1076 = 0.$$

From the last equation we find: $x_1 = 2$, $x_2 = -\frac{538}{51}$.

Check. The first of the found roots is readily checked by substitution into the original equation. Such a check shows that $x_1 = 2$ is a root of Equation (2). The attempt to check the second root using the same method leads to awkward computations. However, it is possible to proceed in a different way. Let us find out whether $x_2 = -\frac{538}{51}$ is a solution of Equation (3). Note that for this value the left-hand side of Equation (3) is positive, as opposed to the right-hand side. Hence $x_2 = -\frac{538}{51}$ is not a root of Equation (3). But Equation (3) is a consequence of Equation (2), and a fortiori x_2 is not a root of Equation (2). Thus, the only root of Equation (2) is $x = 2$.

Example 3. Solve the equation $\sqrt{x+1} - \sqrt[3]{2x-6} = 2$.

Solution. Isolating $\sqrt[3]{2x-6}$, we get: $\sqrt[3]{2x-6} = \sqrt{x+1} - 2$. Cubing both sides of this equation, we get:

$$2x - 6 = (x+1)\sqrt{x+1} - 6(x+1) + 12\sqrt{x+1} - 8.$$

On collecting like terms and isolating the radical, we get the equation $(x + 13)\sqrt{x+1} = 8(x + 1)$, whence we have: $(x + 13)^2(x + 1) = 64(x + 1)^2$, and further, $(x + 1)((x + 13)^2 - 64(x + 1)) = 0$ or $(x + 1)(x^2 - 38x + 105) = 0$.

Thus, the problem is reduced to solving the collection:

$$x + 1 = 0; \quad x^2 - 38x + 105 = 0,$$

whence we find: $x_1 = -1$, $x_2 = 3$, $x_3 = 35$.

Check. Substituting the found values of x into the given equation, we make sure that all of them are its roots.

Example 4. Solve the equation

$$\sqrt[3]{x} + \sqrt[3]{2x-3} = \sqrt[3]{12(x-1)}. \tag{4}$$

Solution. Let us cube both sides of Equation (4) using a somewhat modified formula for the cube of the sum of two numbers, namely,

the formula $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$. We get:

$$x+2x-3+3\sqrt[3]{x(2x-3)}\,(\sqrt[3]{x}+\sqrt[3]{2x-3}) = 12(x-1). \qquad (5)$$

Using Equation (4), we replace the expression $\sqrt[3]{x}+\sqrt[3]{2x-3}$ by $\sqrt[3]{12(x-1)}$. We get:

$$3x-3+3\sqrt[3]{x(2x-3)}\,\sqrt[3]{12(x-1)} = 12(x-1), \qquad (6)$$

or

$$\sqrt[3]{x(2x-3)\,12(x-1)} = 3(x-1).$$

Cubing both sides of the last equation, we get:

$$12x(2x-3)(x-1) = 27(x-1)^3,$$

and further $(x-1)(4x(2x-3)-9(x-1)^2) = 0$, whence we find: $x_1 = 1$, $x_{2,3} = 3$.

Check. Substituting the found values of x into Equation (4), we make sure that they satisfy it.

Remark. Since, when solving Equation (4), we used the operation of cubing both sides of the given equation and, as is known, raising to an odd power does not violate the equivalence of an equation, it might seem that the found solutions require no check. But this is not so. When passing from Equation (5) to Equation (6), we replaced the expression $\sqrt[3]{x}+\sqrt[3]{2x-3}$ by $\sqrt[3]{12(x-1)}$. It is clear that any root of Equation (5) is at the same time a root of Equation (6), but the converse is, generally speaking, not true. Hence, Equation (6) is a consequence of Equation (5), and therefore the check is needed. The following example confirms this thought.

Example 5. Solve the equation $\sqrt[3]{2x-1}+\sqrt[3]{x-1} = 1$.

Solution. We have:

$$(2x-1)+(x-1)+3\sqrt[3]{(2x-1)(x-1)}\,(\sqrt[3]{2x-1}+\sqrt[3]{x-1}) = 1,$$

$$3\sqrt[3]{(2x-1)(x-1)} = 3-3x, \quad (2x-1)(x-1) = (1-x)^3,$$

$$(x-1)((2x-1)+(x-1)^2) = 0,$$

whence $x_1 = 1$, $x_2 = 0$.

Check. Substituting the found values of x into the original equation, we get convinced that the value $x_2 = 0$ does not satisfy the given equation. The latter has the only root $x = 1$.

2. The Method of Introducing New Variables.

Example 6. Solve the equation

$$x^2+3-\sqrt{2x^2-3x+2} = 1.5\,(x+4). \qquad (7)$$

Solution. Isolating the radical and squaring both sides of Equation (7) would lead to an awkward equation. At the same time, we

can easily grasp that Equation (7) is readily reduced to a quadratic equation. Indeed, multiplying both of its sides by 2, we get:

$$2x^2 + 6 - 2\sqrt{2x^2 - 3x + 2} = 3x + 12,$$

and further $2x^2 - 3x + 2 - 2\sqrt{2x^2 - 3x + 2} - 8 = 0$.

Setting $y = \sqrt{2x^2 - 3x + 2}$, we get: $y^2 - 2y - 8 = 0$, whence $y_1 = 4$, $y_2 = -2$. Hence, Equation (7) is equivalent to the following collection of equations:

$$\sqrt{2x^2-3x+2}=4; \quad \sqrt{2x^2-3x+2}=-2.$$

From the first equation of this collection we find: $x_1=\frac{7}{2}$, $x_2=-2$. The second equation has no roots.

Check. Since Equation (7) is equivalent to the equation $\sqrt{2x^2 - 3x + 2} = 4$ (because the second equation of the collection has no solution), the found values can be checked by substituting them into the equation $\sqrt{2x^2 - 3x + 2} = 4$. This substitution shows that both values of x are roots of the indicated equation, and, hence, of Equation (7).

Example 7. Solve the equation

$$2x-5+2\sqrt{x^2-5x}+2\sqrt{x-5}+2\sqrt{x}=48. \qquad (8)$$

Solution. The domain of definition of the equation is $x \geqslant 5$. Under this condition we have: $x \geqslant 0$ and $x - 5 \geqslant 0$, and therefore

$$\sqrt{x^2-5x}=\sqrt{x(x-5)}=\sqrt{x}\sqrt{x-5}.$$

Since $2x = x + x$, Equation (8) may be rewritten as follows:

$$x + x - 5 + 2\sqrt{x}\sqrt{x-5} + 2\sqrt{x-5} + 2\sqrt{x} - 48 = 0,$$

or

$$(\sqrt{x})^2+2\sqrt{x}\sqrt{x-5}+(\sqrt{x-5})^2+2(\sqrt{x-5}+\sqrt{x})-48=0,$$

that is,

$$(\sqrt{x-5}+\sqrt{x})^2+2(\sqrt{x-5}+\sqrt{x})-48=0.$$

Setting $y = \sqrt{x-5} + \sqrt{x}$, we get the quadratic equation $y^2 + 2y - 48 = 0$ wherefrom we find: $y_1 - 6$, $y_2 = -8$. Thus, the problem has been reduced to solving the collection of equations:

$$\sqrt{x-5}+\sqrt{x}=6; \quad \sqrt{x-5}+\sqrt{x}=-8.$$

From the first equation we find $x = \left(\frac{41}{12}\right)^2$, the second equation having no solution.

Check. We can easily show that $x = \left(\frac{41}{12}\right)^2$ is a root of the equation $\sqrt{x-5}+\sqrt{x}=6$. But this equation is equivalent to Equation (8), hence, $x = \left(\frac{41}{12}\right)^2$ is a root of Equation (8) as well.

When solving irrational equations, we sometimes prefer to introduce two new auxiliary variables.

Example 8. Solve the equation

$$\sqrt[4]{1-x}+\sqrt[4]{15+x}=2. \tag{9}$$

Solution. Let us set:

$$\begin{cases} u=\sqrt[4]{1-x} \\ v=\sqrt[4]{15+x}. \end{cases}$$

Then Equation (9) takes the form: $u+v=2$. But to find the values of the new variables, one equation is not sufficient. Raising both sides of each equation to the fourth power, we get

$$\begin{cases} u^4=1-x \\ v^4=15+x. \end{cases}$$

We then add together the equations of the last system: $u^4 + v^4 = 16$.

Thus, for finding u, v we have the following symmetric system of equations:

$$\begin{cases} u+v=2 \\ u^4+v^4=16. \end{cases}$$

Solving this system (see Item 4 of Sec. 10) we find (confining ourselves to real solutions):

$$\begin{cases} u_1=0 \\ v_1=2 \end{cases}; \quad \begin{cases} u_2=2 \\ v_2=0. \end{cases}$$

The problem has been reduced to solving the collection of the systems:

$$\begin{cases} \sqrt{1-x}=0 \\ \sqrt{15+x}=2 \end{cases}; \quad \begin{cases} \sqrt[4]{1-x}=2 \\ \sqrt{15+x}=0. \end{cases}$$

Solving this collection, we find: $x_1 = 1$, $x_2 = -15$.

Check (the simplest way to carry out the check is to substitute the found values into the original equation). The check convinces us that both found values of x are roots of the original equation.

Remark. This method might also be used for solving some of the equations considered above. Thus, when solving the equation $\sqrt{x+1}-\sqrt[3]{2x-6}=2$ (see Example 3), we could set $\begin{cases} u=\sqrt{x+1} \\ v=\sqrt[3]{2x+6} \end{cases}$ and arrive at the system of equations

$$\begin{cases} u-v=2 \\ 2u^2-v^3=8. \end{cases}$$

Example 9. Solve the equation

$$\sqrt{\frac{\sqrt{x^2+28^2}+x}{x}}-\sqrt{x\sqrt{x^2+28^2}-x^2}=3. \tag{10}$$

Solution. Setting

$$\begin{cases} u=\sqrt{\dfrac{\sqrt{x^2+28^2}+x}{x}} \\ v=\sqrt{x\sqrt{x^2+28^2}-x^2}, \end{cases} \tag{11}$$

we get the equation $u-v=3$. Multiplying together the right-hand sides of the equations of System (11), we get:

$$\sqrt{\frac{\sqrt{x^2+28^2}+x}{x}}\sqrt{x\sqrt{x^2+28^2}-x^2}$$
$$=\sqrt{\frac{\sqrt{x^2+28^2}+x}{x}x\left(\sqrt{x^2+28^2}-x\right)}=\sqrt{\left(\sqrt{x^2+28^2}\right)^2-x^2}=28.$$

The obtained result leads to another equation in new variables: $uv=28$. Solving the system

$$\begin{cases} u-v=3 \\ u\times v=28, \end{cases}$$

we find:

$$\begin{cases} u_1=7 \\ v_1=4 \end{cases}; \quad \begin{cases} u_2=-4 \\ v_2=-7. \end{cases}$$

Thus, we come to a collection of systems of equations. From this collection we take only the system corresponding to positive values of u_1 and v_1 (the system corresponding to negative u_2,, v_2 a fortiori has no solution, therefore we omit it):

$$\begin{cases} \sqrt{\dfrac{\sqrt{x^2+28^2}+x}{x}}=7 \\ \sqrt{x\sqrt{x^2+28^2}-x^2}=4. \end{cases} \tag{12}$$

We now solve the second equation of System (12). Squaring both sides of this equation, we get: $x\sqrt{x^2+28^2}-x^2=16$, and further

$$x\sqrt{x^2+28^2}=x^2+16. \qquad (13)$$

Let us now square both sides of Equation (13):

$$x^2(x^2+28^2)=(x^2+16)^2, \qquad (14)$$

and further: $752x^2-256=0$.

From the last equation we find:

$$x_1=\frac{4\sqrt{47}}{47},\ x_2=-\frac{4\sqrt{47}}{47}$$

Check. It is clear that x_2 does not satisfy Equation (13) and, hence, the second equation of System (12). Let us check x_1. Since for $x>0$ Equations (14), (13), and the second equation of System (12) are equivalent, $x=\frac{4\sqrt{47}}{47}$ is a solution of the second equation of System (12). Now we must get convinced that the found value of x_1 also satisfies the first equation of System (12) (only in this case we may regard this value as the solution of System (12)). Let us reduce this equation to a simpler equivalent. We have:

$$\sqrt{\frac{\sqrt{x^2+28^2}+x}{x}}=7,\ \sqrt{x^2+28^2}+x=49x,\ \sqrt{x^2+28^2}=48x,$$

$$x^2+28^2=48^2x^2,\ (48^2-1)\,x^2=28^2,$$

whence $x^2=\frac{16}{47}$.

The value of x_1 satisfies the last equation and at the same time the first equation of System (12).

Thus, $x=\frac{4\sqrt{47}}{47}$ is the solution of System (12), and, hence, of Equation (10).

Example 10. Solve the equation

$$\sqrt[5]{(x-2)(x-32)}-\sqrt[4]{(x-1)(x-33)}=1. \qquad (15)$$

Solution. Let us set

$$\begin{cases} u=\sqrt[5]{(x-2)(x-32)} \\ v=\sqrt[4]{(x-1)(x-33)}. \end{cases} \qquad (16)$$

Then Equation (15) takes the form: $u-v=1$. To obtain the second equation in new variables u and v, let us raise both sides of the first equation of System (16) to the fifth power and those of the

second equation to the fourth. We get:

$$\begin{cases} u^5 = x^2 - 34x + 64 \\ v^4 = x^2 - 34x + 33, \end{cases}$$

whence $u^5 - v^4 = 31$. Thus, for finding u and v, we have the following system of equations:

$$\begin{cases} u - v = 1 \\ u^5 - v^4 = 31 \end{cases} \text{ or } \begin{cases} v = u - 1 \\ u^5 - (u-1)^4 = 31, \end{cases}$$

whence

$$\begin{cases} v = u - 1 \\ u^5 - u^4 + 4u^3 - 6u^2 + 4u - 32 = 0. \end{cases} \tag{17}$$

From the second equation of System (17) we find: $u_1 = 2$. Dividing the polynomial $u^5 - u^4 + 4u^3 - 6u^2 + 4u - 32$ by the binomial $u - 2$, we get: $u^4 + u^3 + 6u^2 + 6u + 16$.

Thus, System (17) is equivalent to the collection of systems:

$$\begin{cases} v = u - 1 \\ u - 2 = 0 \end{cases}; \quad \begin{cases} v = u - 1 \\ u^4 + u^3 + 6u^2 + 6u + 16 = 0. \end{cases}$$

From the first system we find: $u_1 = 2$, $v_1 = 1$.

The second system is more complicated. In the process of solving this system the following should be taken into consideration. Since $\sqrt[4]{(x-1)(x-33)} = v$, $v \geqslant 0$.

Since $u - v = 1$, $u = v + 1$, and, consequently, $u \geqslant 1$. It is obvious that the equation

$$u^4 + u^3 + 6u^2 + 6u + 16 = 0$$

has no solutions which satisfy the inequality $u \geqslant 1$.

Thus, $\begin{cases} u_1 = 2 \\ v_1 = 1 \end{cases}$ is the only solution of System (17), and it remains to solve the following system:

$$\begin{cases} \sqrt[5]{(x-2)(x-32)} = 2 \\ \sqrt[4]{(x-1)(x-33)} = 1. \end{cases}$$

We have:

$$\begin{cases} \sqrt[5]{x^2 - 34x + 64} = 2 \\ \sqrt[4]{x^2 - 34x + 33} = 1. \end{cases}$$

If we set $y = x^2 - 34x + 33$, then the system takes the form:

$$\begin{cases} \sqrt[5]{y + 31} = 2 \\ \sqrt[4]{y} = 1. \end{cases}$$

From this system we find: $y = 1$. Then $x^2 - 34x + 33 = 1$, whence $x_{1,2} = 17 \pm \sqrt{257}$.

Check. Analysing the transformations carried in the course of solution (all of them are equivalent—make sure that this is so), we conclude that the found values of x are roots of Equation (15).

Artificial Methods of Solving Irrational Equations.

Example 11. Solve the equation

$$\sqrt{2x^2 + 3x + 5} + \sqrt{2x^2 - 3x + 5} = 3x. \qquad (18)$$

Solution. We multiply both sides of the given equation by the function $\varphi(x) = \sqrt{2x^2 + 3x + 5} - \sqrt{2x^2 - 3x + 5}$, conjugate to $\sqrt{2x^2 + 3x + 5} + \sqrt{2x^2 - 3x + 5}$.

Since $(\sqrt{2x^2 + 3x + 5} + \sqrt{2x^2 - 3x + 5})(\sqrt{2x^2 + 3x + 5} - \sqrt{2x^2 - 3x + 5}) = (2x^2 + 3x + 5) - (2x^2 - 3x + 5) = 6x$, Equation (18) takes the form:

$$6x = 3x(\sqrt{2x^2 + 3x + 5} - \sqrt{2x^2 - 3x + 5}),$$

or

$$x(\sqrt{2x^2 + 3x + 5} - \sqrt{2x^2 - 3x + 5} - 2) = 0. \qquad (19)$$

As is easily seen, $x_1 = 0$ is one of the roots of Equation (19). It remains to solve the equation

$$\sqrt{2x^2 + 3x + 5} - \sqrt{2x^2 - 3x + 5} = 2. \qquad (20)$$

Adding together Equations (18) and (20), we get the consequence:

$$2\sqrt{2x^2 + 3x + 5} = 3x + 2. \qquad (21)$$

Solving Equation (21) by squaring we get:

$$8x^2 + 12x + 20 = 9x^2 + 12x + 4,$$

and further: $x^2 = 16$, whence $x_2 = 4$, $x_3 = -4$.

Check. Substituting the found values $x_1 = 0$, $x_2 = 4$, $x_3 = -4$ into Equation (18), we see that it is satisfied only by the value $x_2 = 4$. Thus, $x = 4$ is the only root of Equation (18).

Example 12. Solve the equation

$$\sqrt[4]{x - 1} + 2\sqrt[3]{3x + 2} = 4 + \sqrt{3 - x}. \qquad (22)$$

Solution. In this case, neither of the above methods proves to be successful. Let us make an attempt to find some solution of the given equation using the trial method. The domain of definition of the

equation is given by the system of inequalities: $\begin{cases} x - 1 \geqslant 0 \\ 3 - x \geqslant 0, \end{cases}$ whence we get: $1 \leqslant x \leqslant 3$. Hence, solutions should be sought for only in this interval. Trying the integral values of x from the indicated interval, we find that $x = 2$ is a root of the given equation. If we now prove that the original equation has no other roots, the solution of the equation will be thereby completed.

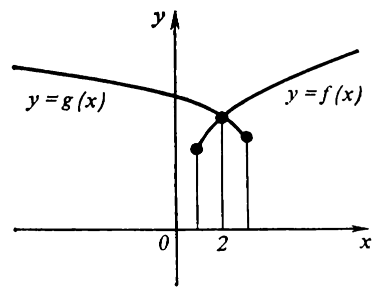

Fig. 4

On the interval $[1, 3]$ the function $f(x) = \sqrt[4]{x - 1} + 2\sqrt[3]{3x + 2}$ is increasing, while the function $g(x) = 4 + \sqrt{3 - x}$ is decreasing. But in this case, if the equation $f(x) = g(x)$, has, in general, roots, then there is only one root (see Fig. 4). Hence, $x = 2$ is the only root of Equation (22).

4. Systems of Irrational Equations.

Example 13. Solve the system of equations

$$\begin{cases} \sqrt{\dfrac{3x-2y}{2x}} + \sqrt{\dfrac{2x}{3x-2y}} = 2 \\ 4y^2 - 1 = 3y(x-1). \end{cases} \tag{23}$$

Solution. Let us set $u = \sqrt{\dfrac{3x-2y}{2x}}$. Then the first equation of the system takes the form: $u + \dfrac{1}{u} = 2$, whence we find: $u = 1$.

Thus, the solution of System (23) is reduced to solving the following system:

$$\begin{cases} \sqrt{\dfrac{3x-2y}{2x}} = 1 \\ 4y^2 - 1 = 3y(x-1). \end{cases} \tag{24}$$

Squaring both sides of the first equation of System (24) and getting rid of the denominator, we obtain:

$$\begin{cases} 3x - 2y = 2x \\ 4y^2 - 1 = 3y(x-1), \end{cases} \tag{25}$$

whence we find: $\begin{cases} x_1 = 2 \\ y_1 = 1 \end{cases}$; $\begin{cases} x_2 = 1 \\ y_2 = \dfrac{1}{2}. \end{cases}$

Check. System (23) is equivalent to System (24). Since for $x \neq 0$ and $3x \neq 2y$ both sides of the first equation of System (24) are non-

negative, System (24) is equivalent to System (25). Thus, the solutions of System (25) are also solutions of System (23). Hence, the pairs (2, 1) and $\left(1, \frac{1}{2}\right)$ are solutions of System (23).

Example 14. Solve the system of equations

$$\begin{cases} 2\sqrt{x-y}+\sqrt[4]{x+2y}=4 \\ \sqrt[8]{(x-y)^4(x+2y)^2}=2. \end{cases}$$

Solution. Setting $\begin{cases} u=\sqrt{x-y} \\ v=\sqrt[4]{x+2y}, \end{cases}$ we get the system of equations: $\begin{cases} 2u+v=4 \\ uv=2, \end{cases}$ from which we find: $u=1$, $v=2$.

Thus, the problem has been reduced to solving the following system:

$$\begin{cases} \sqrt{x-y}=1 \\ \sqrt[4]{x+2y}=2 \end{cases} \text{ or } \begin{cases} x-y=1 \\ x+2y=16, \end{cases}$$

whence $x=6$, $y=5$.

It is easy to check that the found solution of the last system is also the solution of the original system. Thus, the pair (6, 5) is the solution of the given system of equations.

EXERCISES

In Problems 609 through 678, solve the indicated equations:

609. $\sqrt{x-1}+\sqrt{2-x}=3$. 610. $\sqrt{x+1}-\sqrt{9-x}=\sqrt{2x-12}$.

611. $\sqrt{2x+5}+\sqrt{5x+6}=\sqrt{12x+25}$.

612. $\sqrt{x}-\sqrt{x+1}+\sqrt{x+9}-\sqrt{x+4}=0$. 613. $\sqrt{2x+\sqrt{6x^2+1}}=x+1$.

614. $(1+x^2)\sqrt{1+x^2}=x^2-1$. 615. $\sqrt{x^2+1}+\sqrt{x^2-8}=3$.

616. $1-x=\sqrt{1-\sqrt{4x^2-7x^4}}$. 617. $\sqrt{1-\sqrt{x^4-x^2}}=x-1$.

618. $\sqrt{7+\sqrt[3]{x^2+7}}=3$. 619. $\sqrt{5+\sqrt[3]{x}}+\sqrt{5-\sqrt[3]{x}}=\sqrt[3]{x}$.

620. $\sqrt{3x^2-2x+15}+\sqrt{3x^2-2x+8}=7$.

621. $\sqrt{3x^2+5x+8}-\sqrt{3x^2+5x+1}=1$.

622. $\sqrt{x^2-3x+3}+\sqrt{x^2-3x+6}=3$. 623. $x^2+\sqrt{x^2+20}=22$.

624. $\sqrt{x^3+8}+\sqrt[4]{x^3+8}=6$. 625. $\sqrt{x\sqrt[5]{x}}-\sqrt[5]{x\sqrt{x}}=56$.

626. $\sqrt[7]{\frac{5-x}{x+3}}+\sqrt[7]{\frac{x+3}{5-x}}=2.$ 627. $\sqrt[4]{\frac{2-x}{3+x}}+\sqrt[4]{\frac{3+x}{2-x}}=2.$

628. $x\sqrt{x^2+15}-\sqrt{x}\sqrt[4]{x^2+15}=2.$ 629. $x^2-4x-6=\sqrt{2x^2-8x+12}.$

630. $(x+4)(x+1)-3\sqrt{x^2+5x+2}=6.$ 631. $\sqrt{x^2-3x+5}+x^2=3x+7.$

632. $x^2-3x-5\sqrt{9x^2+x-2}=2.75-\frac{28}{9}x.$ 633. $x+\sqrt[3]{x}-2=0.$

634. $x-4\sqrt[3]{x^2}+\sqrt[3]{x}+6=0.$ 635. $4x-3\sqrt[3]{x}-1=0.$

636. $x^{10}-x^5-2\sqrt{x^5}+2=0.$ 637. $\sqrt{x^2+x+4}+\sqrt{x^2+x+1}=\sqrt{2x^2+2x+9}.$

638. $\sqrt{x^2+x+7}+\sqrt{x^2+x+2}=\sqrt{3x^2+3x+19}.$

639. $\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}=x-1.$

640. $\sqrt{5-\sqrt{x+1}+\sqrt{2x^2+x+3}}=1.$

641. $\sqrt{x+8+2\sqrt{x+7}}+\sqrt{x+1-\sqrt{x+7}}=4.$

642. $\sqrt{x-2+\sqrt{2x-5}}+\sqrt{x+2+3\sqrt{2x-5}}=7\sqrt{2}.$

643. $\sqrt[3]{2x(4x^2+3)-1-12x^2}+x=x^2-11.$

644. $\sqrt{x^2-4x+3}+\sqrt{-x^2+3x-2}=\sqrt{x^2-x}.$

645. $\sqrt{x^2-x-1}+\sqrt{x^2+x+3}=\sqrt{2x^2+8}$ (find the positive solutions).

646. $\sqrt{x^2+x-2}+\sqrt{x^2+2x-3}=\sqrt{x^2-3x+2}.$ 647. $\sqrt[3]{x+24}+\sqrt{12-x}=6.$

648. $\sqrt[6]{1.5}\sqrt[3]{x-\frac{1}{x}}-\sqrt{x-\frac{1}{x}}=0.$

649. $\sqrt{x+2}-\sqrt[3]{3x+2}=0.$ 650. $\sqrt{x}+\sqrt[3]{x-1}=1.$

651. $\sqrt[3]{2-x}=1-\sqrt{x-1}.$ 652. $\sqrt[3]{x+7}+\sqrt[3]{28-x}=5.$

653. $\sqrt[3]{x^2-1}+\sqrt[3]{x^2+18}=5.$ 654. $\sqrt[3]{x+1}+\sqrt[3]{x+2}+\sqrt[3]{x+3}=0.$

655. $\sqrt[3]{x}+\sqrt[3]{x-16}=\sqrt[3]{x-8}.$ 656. $\sqrt{9-\sqrt{x+1}}+\sqrt[3]{7+\sqrt{x+1}}=4.$

657. $\sqrt[3]{54+\sqrt{x}}+\sqrt[3]{54-\sqrt{x}}=\sqrt[3]{18}.$

658. $\sqrt[4]{78+\sqrt[3]{24+\sqrt{x}}}-\sqrt[4]{84-\sqrt[3]{30-\sqrt{x}}}=0.$

659. $\sqrt{\frac{20+x}{x}}+\sqrt{\frac{20-x}{x}}=\sqrt{6}.$ 660. $\sqrt{x^3+x^2-1}+\sqrt{x^3+x^2+2}=3.$

661. $x\sqrt[3]{35-x^3}(x+\sqrt[3]{35-x^3})=30.$ 662. $x+\sqrt{17-x^2}+x\sqrt{17-x^2}=9.$

663. $\sqrt[4]{x-2}+\sqrt[4]{6-x}=\sqrt{2}.$ 664. $\sqrt[4]{77+x}+\sqrt[4]{20-x}=5.$

665. $\sqrt[4]{97-x}+\sqrt[4]{x}=5.$ 666. $\sqrt{6-x}+\sqrt{x-2}+2\sqrt[4]{(6-x)(x-2)}=2.$

667. $\sqrt[5]{33-x}+\sqrt[5]{x}=3.$ 668. $\sqrt[5]{(x-2)(x-32)}-\sqrt[5]{(x-1)(x-33)}=1.$

669. $\sqrt{\dfrac{\sqrt{x^2+66^2}+x}{x}}-\sqrt{x\sqrt{x^2+66^2}-x^2}=5.$

670. $4(\sqrt{1+x}-1)(\sqrt{1-x}+1)=x.$

671. $x+\sqrt{x}+\sqrt{x+2}+\sqrt{x^2+2x}=3.$

672. $\sqrt{x^3-4x^2+x+15}+\sqrt{x^3-4x^2-x+13}=x+1.$

673. $\sqrt{(x-1)(x-2)}+\sqrt{(x-3)(x-4)}=\sqrt{2}.$

674. $\sqrt[3]{4-4x+x^2}+\sqrt[3]{49+14x+x^2}=3+\sqrt[3]{14-5x-x^2}.$

675. $\sqrt{x-1}+\sqrt{x+3}+2\sqrt{(x-1)(x+3)}=4-2x.$

676. $\sqrt{2x+3}+\sqrt{x+1}=3x+2\sqrt{2x^2+5x+3}-16.$

677. $\dfrac{\sqrt{x+4}+\sqrt{x-4}}{2}=x+\sqrt{x^2-16}-6.$

678. $\sqrt{x}+\sqrt{x+7}+2\sqrt{x^2+7x}=35-2x.$

In Problems 679 through 700, solve the given systems of equations:

679. $\begin{cases}\sqrt{\dfrac{y}{x}}-2\sqrt{\dfrac{x}{y}}=1\\ \sqrt{5x+y}+\sqrt{5x-y}=4.\end{cases}$

680. $\begin{cases}\sqrt{\dfrac{x^3}{y}}-\sqrt{\dfrac{y^3}{x}}=\dfrac{65}{6}\\ x-y=5.\end{cases}$

681. $\begin{cases}y^2+\sqrt{3y^2-2x+3}=\dfrac{2}{3}x+5\\ 3x-2y=5.\end{cases}$

682. $\begin{cases}5\sqrt[3]{x-2y}+3\sqrt[3]{x+y}=13\\ 3\sqrt[3]{x-2y}-4\sqrt[3]{x+y}=2.\end{cases}$

683. $\begin{cases}\dfrac{\sqrt[3]{2x+y}}{y}+\dfrac{\sqrt[3]{2x+y}}{2x}=\dfrac{81}{182}\\ \dfrac{\sqrt[3]{2x-y}}{y}-\dfrac{\sqrt[3]{2x-y}}{2x}=\dfrac{1}{182}.\end{cases}$

684. $\begin{cases}\sqrt{2x-y+11}-\sqrt{3x+y-9}=3\\ \sqrt[4]{2x-y+11}+\sqrt[4]{3x+y-9}=3.\end{cases}$

685. $\begin{cases}\sqrt{x+y}+\sqrt[3]{x-y}=6\\ \sqrt[6]{(x+y)^3(x-y)^2}=8.\end{cases}$

686. $\begin{cases}x+y+\sqrt{xy}=14\\ x^2+y^2+xy=84.\end{cases}$

687. $\begin{cases}\dfrac{1}{\sqrt{x}}+\dfrac{1}{\sqrt{y}}=\dfrac{4}{3}\\ xy=9.\end{cases}$

688. $\begin{cases}x\sqrt{y}+y\sqrt{x}=6\\ x^2y+y^2x=20.\end{cases}$

689. $\begin{cases}\sqrt[3]{x}+\sqrt[3]{y}=3\\ xy=8.\end{cases}$

690. $\begin{cases}\sqrt[3]{x}+\sqrt[3]{y}=3\\ \sqrt[3]{x^2}-\sqrt[3]{xy}+\sqrt[3]{y^2}=3.\end{cases}$

691. $\begin{cases}x^2+x\sqrt[3]{xy^2}=80\\ y^2+y\sqrt[3]{x^2y}=5.\end{cases}$

692. $\begin{cases}\sqrt[3]{\dfrac{y+1}{x}}-2\sqrt[3]{\dfrac{x}{y+1}}=1\\ \sqrt{x+y+1}+\sqrt{x-y+10}=5.\end{cases}$

693. $\begin{cases}\sqrt{x}+\sqrt{y+1}=1\\ \sqrt{x+1}+\sqrt{y}=1.\end{cases}$

694. $\begin{cases} \sqrt{\dfrac{20y}{x}} = \sqrt{x+y} + \sqrt{x-y} \\ \sqrt{\dfrac{16x}{5y}} = \sqrt{x+y} - \sqrt{x-y}. \end{cases}$

695. $\begin{cases} \sqrt[3]{x+2y} + \sqrt[3]{x-y+2} = 3 \\ 2x+y = 7. \end{cases}$

696. $\begin{cases} \sqrt{x+\sqrt{y}} + \sqrt{x-\sqrt{y}} = 2 \\ \sqrt{y+\sqrt{x}} - \sqrt{y-\sqrt{x}} = 1. \end{cases}$

697. $\begin{cases} \sqrt{xy} + \sqrt{yz} = 9 \\ \sqrt{yz} + \sqrt{zx} = 5 \\ \sqrt{xz} + \sqrt{xy} = 8. \end{cases}$

698. $\begin{cases} \sqrt{x+y} + \sqrt{y+z} = 3 \\ \sqrt{y+z} + \sqrt{z+x} = 5 \\ \sqrt{z+x} + \sqrt{x+y} = 4. \end{cases}$

699. $\begin{cases} \sqrt{x-4} + \sqrt{y} + \sqrt{z+4} = 6 \\ 2\sqrt{x-4} - \sqrt{y} - 4\sqrt{z+4} = -12 \\ x+y+z = 14. \end{cases}$

700. $\begin{cases} \sqrt{4x+y-3z+7} = 2 \\ \sqrt[3]{2y+5x+z+25.5} = 3 \\ \sqrt{y+z} - \sqrt{6x} = 0. \end{cases}$

SEC. 13. EXPONENTIAL EQUATIONS

When solving exponential equations we use two basic methods: (1) replacing the equation $a^{f(x)} = a^{g(x)}$ by the equation $f(x) = g(x)$; (2) introducing new variables. Sometimes, we have to use artificial methods.

1. Exponential Equations. Consider equations of the form $a^{f(x)} = a^{g(x)}$, where $a > 0$ and $a \neq 1$ and equations which can be reduced to them. The solution of such equations is based on the following theorem.

Theorem. *If $a > 0$ and $a \neq 1$, then the equation $a^{f(x)} = a^{g(x)}$ is equivalent to the equation $f(x) = g(x)$.*

Example 1. Solve the equation $2^{x^2-2x} = 2^{3x-6}$.

Solution. The given equation is equivalent to the equation $x^2 - 2x = 3x - 6$, and therefore the roots of the last equation $x_1 = 2$ and $x_2 = 3$ are also roots of the original equation.

Example 2. Solve the equation $\dfrac{0.2^{x-0.5}}{\sqrt{5}} = 5 \times 0.04^{x-1}$.

Solution. We reduce all the powers to the same base $\frac{1}{5}$:

$$\left(\frac{1}{5}\right)^{x-0.5} \times \left(\frac{1}{5}\right)^{0.5} = \left(\frac{1}{5}\right)^{-1} \times \left[\left(\frac{1}{5}\right)^{2}\right]^{x-1}.$$

Further, we have: $\left(\frac{1}{5}\right)^{x} = \left(\frac{1}{5}\right)^{2x-3}$.

The last equation is equivalent to the equation $x = 2x - 3$, wherefrom we find: $x = 3$. Thus, $x = 3$ is the only root of the given equation.

Example 3. Solve the equation

$$3^{x^2-4} = 5^{2x}. \qquad (1)$$

Solution. Since $5 = 3^{\log_3 5}$, Equation (1) can be transformed to $3^{x^2-4} = (3^{\log_3 5})^{2x}$.

This equation is equivalent to the following:

$$x^2 - 4 = 2x \log_3 5. \qquad (2)$$

The roots of the quadratic equation (2) and, at the same time, of the given exponential equation (1) are: $x_{1,2} = \log_3 5 \pm \sqrt{\log_3^2 5 + 4}$.

Example 4. Solve the equation

$$5^{1+2x} + 6^{1+x} = 30 + 150^x. \qquad (3)$$

Solution. Since $5^{1+2x} = 5 \times 25^x$, $6^{1+x} = 6 \times 6^x$ and $150^x = 6^x \times 25^x$, Equation (3) can be transformed to:

$$5 \times 25^x + 6 \times 6^x - 6^x \times 25^x - 30 = 0,$$

and further $5\,(25^x - 6) - 6^x\,(25^x - 6) = 0$, $(25^x - 6)\,(5 - 6^x) = 0$.

The last equation is reduced to the collection of equations

$$25^x - 6 = 0; \quad 5 - 6^x = 0,$$

which has the following solutions: $x_1 = \log_{25} 6$, $x_2 = \log_6 5$.

The found values of x are roots of Equation (3).

Example 5. Solve the equation

$$4^x + 2^{x+1} - 24 = 0. \qquad (4)$$

Solution. Let us apply the method of introducing new variables. Since $4^x = (2^2)^x = (2^x)^2$ and $2^{x+1} = 2 \times 2^x$, Equation (4) can be rewritten in the following way:

$$(2^x)^2 + 2 \times 2^x - 24 = 0.$$

Setting $u = 2^x$, we get the quadratic equation $u^2 + 2u - 24 = 0$, whose roots are: $u_1 = 4$ and $u_2 = -6$. Therefore the problem is reduced to solving the collection of equations: $2^x = 4$; $2^x = -6$.

From the first equation of this collection we get: $x = 2$. The second equation of the collection has no solutions since $2^x > 0$ for any values of x. Thus, the root of Equation (4) is $x = 2$.

Example 6. Solve the equation

$$2^x + (0.5)^{2x-3} - 6\,(0.5)^x = 1.$$

Solution. Since $(0.5)^{2x-3}=2^{3-2x}=\frac{8}{2^{2x}}$ and $6\,(0.5)^x=\frac{6}{2^x}$, we have:

$$2^x+\frac{8}{2^{2x}}-\frac{6}{2^x}-1=0.$$

Setting $u = 2^x$, we get: $u+\frac{8}{u^2}-\frac{6}{u}-1=0$, and further $u^3-u^2-6u+8=0$, that is, $(u-2)(u^2+u-4)=0$.

The last equation has three roots: $u_1=2$, $u_2=\frac{-1+\sqrt{17}}{2}$, $u_3=\frac{-1-\sqrt{17}}{2}$.

Now, the problem is reduced to solving the collection of equations:

$$2^x=2;\ 2^x=\frac{\sqrt{17}-1}{2};\ 2^x=\frac{-1-\sqrt{17}}{2}.$$

From the first equation we find: $x_1=1$, from the second: $-x_2=\log_2\frac{\sqrt{17}-1}{2}$. The third equation has no solution since $\frac{-1-\sqrt{17}}{2}<0$ and $2^x>0$ for $x\in R$.

Hence, the original equation has the following roots: $x_1=1$ and $x_2=\log_2\frac{\sqrt{17}-1}{2}$.

Example 7. Solve the equation

$$6\times 3^{2x}-13\times 6^x+6\times 2^{2x}=0. \tag{5}$$

Solution. Since $6^x=3^x\times 2^x$, we have:

$$6\times 3^{2x}-13\times 3^x\times 2^x+6\times 2^{2x}=0.$$

Setting $u=3^x$, $v=2^x$, we get the equation:

$$6u^2-13uv+6v^2=0, \tag{6}$$

which is a homogeneous equation of the second degree in two variables u and v. Since $v=2^x$ does not vanish for any values of x, dividing both sides of Equation (6) by v^2, we get the following equation which is equivalent to (6):

$$6\left(\frac{u}{v}\right)^2-13\frac{u}{v}+6=0.$$

Setting $z=\frac{u}{v}$, we get: $6z^2-13z+6=0$, whence $z_1=\frac{3}{2}$, $z_2=\frac{2}{3}$.

Taking into account that $z = \frac{u}{v} = \left(\frac{3}{2}\right)^x$, we write the collection of equations:

$$\left(\frac{3}{2}\right)^x = \frac{3}{2}; \ \left(\frac{3}{2}\right)^x = \frac{2}{3},$$

wherefrom we find: $x_1 = 1$, $x_2 = -1$.

Hence, Equation (5) has two roots: $x_1 = 1$, $x_2 = -1$.

Example 8. Solve the equation

$$\left(\frac{3}{5}\right)^x + \frac{7}{5} = 2^x. \qquad (7)$$

Solution. No method of those considered in the previous examples is suitable for solving this equation. Let us try to find a solution of Equation (7) by trial and error method which readily yields. In this case: $x_1 = 1$. Of course, we are not sure that the equation is already solved since it may have some other roots. Let us prove that there are no other roots.

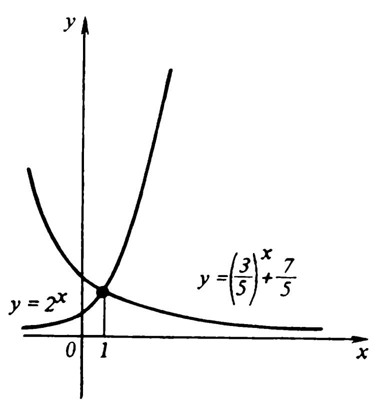

Fig. 5

The function $\left(\frac{3}{5}\right)^x + \frac{7}{5}$ decreases, and the function 2^x increases throughout the number line. Hence, Equation (7) cannot have more than one root (see Fig. 5). Thus, $x = 1$ is the only root of Equation (7).

2. Exponential-power Equations. These are equations of the form $(f(x))^{g(x)} = (f(x))^{h(x)}$. If it is known that $f(x) > 0$ and $f(x) \neq 1$, then this equation, like an exponential one, is solved by equating the exponents: $g(x) = h(x)$. If the possibility of relations $f(x) \leqslant 0$ or $f(x) = 1$ is not excluded by the hypothesis, we have to consider several cases as in the following example.

Example 9. Solve the equation

$$(x^2 + x - 57)^{3x^2+3} = (x^2 + x - 57)^{10x}. \qquad (8)$$

Solution. When solving the given exponential-power equation, we have to consider four cases:

(1) $x^2 + x - 57 = 1$, that is, $x^2 + x - 58 = 0$.

In this case Equation (8) takes the form $1^{3x^2+3} = 1^{10x}$, i.e. $1 = 1$. Hence, the roots of the equation $x^2 + x - 58 = 0$ are also roots of Equation (8). From the equation $x^2 + x - 58 = 0$ we find:

$x_{1,2} = \frac{-1 \pm \sqrt{233}}{2}$.

(2) $x^2 + x - 57 = -1$, that is, $x^2 + x - 56 = 0$. In this case Equation (8) takes the form

$$(-1)^{3x^2+3} = (-1)^{10x}. \qquad (9)$$

Equation (9) can be satisfied only by those values of x for which $3x^2 + 3$ and $10x$ are integers (since the negative number (-1) can be raised only to an integer power) of equal parity (both even or both odd).

From the equation $x^2 + x - 56 = 0$ we find: $x_1 = -8$, $x_2 = 7$. The value $x_1 = -8$ does not satisfy Equation (9), while the value $x_2 = 7$ satisfies this equation. Hence, $x = 7$ is a root of Equation (8).

(3) $x^2 + x - 57 = 0$. In this case Equation (8) takes the form

$$0^{3x^2+3} = 0^{10x}. \qquad (10)$$

Equation (10) can be satisfied only by those values of x for which $3x^2 + 3 > 0$ (this is true for all x's) and $10x > 0$; in this case Equation (10) takes the form $0 = 0$ (let us recall that the expression 0^r has sense only for $r > 0$).

From the equation $x^2 + x - 57 = 0$ we find:

$$x_{1,2} = \frac{-1 \pm \sqrt{229}}{2}.$$

The value $x_1 = \frac{-1-\sqrt{229}}{2}$ does not satisfy the condition $10x > 0$, while $x_2 = \frac{-1+\sqrt{229}}{2}$ does. Hence, $x = \frac{-1+\sqrt{229}}{2}$ is a root of Equation (8).

(4) If $x^2 + x - 57 > 0$ and $x^2 + x - 57 \neq 1$, then from Equation (8) we conclude that $3x^2 + 3 = 10x$, whence we find: $x_1 = 3$, $x_2 = \frac{1}{3}$. Both of these values must be checked by substituting into Equation (8). For $x = 3$ we get: $(-45)^{30} = (-45)^{30}$ which is a true equality.

For $x = \frac{1}{3}$ Equation (8) takes the form

$$\left(\frac{4}{9} - 57\right)^{\frac{10}{3}} = \left(\frac{4}{9} - 57\right)^{\frac{10}{3}}.$$

This has no sense (a negative number is raised to a fractional power). Hence, only $x = 3$ is a root of Equation (8).

Summing up, we conclude that Equation (8) has five roots: $x_{1,2} = \frac{-1 \pm \sqrt{233}}{2}$, $x_3 = 7$, $x_4 = \frac{-1+\sqrt{229}}{2}$, $x_5 = 3$.

EXERCISES

In Problems 701 through 735, solve the given equations:

701. $\left(\frac{2}{3}\right)^x \times \left(\frac{9}{8}\right)^x = \frac{27}{64}$. 702. $2^{x^2} \times 5^{x^2} = 0.001 \times (10^{3-x})^2$.

703. $\left(\frac{3}{4}\right)^{x-1} \times \left(\frac{4}{3}\right)^{\frac{1}{x}} = \frac{9}{16}$. 704. $(0.6)^x \times \left(\frac{25}{9}\right)^{x^2-12} = \left(\frac{27}{125}\right)^3$.

705. $\sqrt{2^x \sqrt[3]{4^x \times 0.125^{\frac{1}{x}}}} = 4\sqrt[3]{2}$. 706. $10^x - 5^{x-1} \times 2^{x-2} = 950$.

707. $2^{3x} \times 3^x - 2^{3x-1} \times 3^{x+1} = -288$. 708. $2 \times 7^{3x} - 5 \times 49^{3x} + 3 = 0$.

709. $3 \times 5^{2x-1} - 2 \times 5^{x-1} = 0.2$. 710. $9^{x^2-1} - 36 \times 3^{x^2-3} + 3 = 0$.

711. $\frac{2^x + 10}{4} = \frac{9}{2^{x-2}}$. 712. $3^{3x+1} - 4 \times 27^{x-1} + 9^{1.5x-1} - 80 = 0$.

713. $4^{\sqrt{x+1}} - 2^{\sqrt{x+1}+2} = 0$. 714. $2^{x+\sqrt{x^2-4}} - 5(\sqrt{2})^{x-2+\sqrt{x^2-4}} - 6 = 0$.

715. $5^{2x} - 7^x - 35 \times 5^{2x} - 35 \times 7^x = 0$. 716. $4^x - 3^{x-0.5} = 3^{x+0.5} - 2^{2x-1}$.

717. $(2+\sqrt{3})^x + (2-\sqrt{3})^x = 4$. 718. $4^x + 6^x = 2 \times 9^x$.

719. $2 \times 4^x + 25^{x+1} = 15 \times 10^x$. 720. $16^x + 36^x = 2 \times 81^x$.

721. $56 \times 4^{x-1} - 53 \times 14^x + 2 \times 49^{x+0.5} = 0$.

722. $2^{2x} \times 9^x - 2 \times 6^{3x-1} + 4^{2x-1} \times 3^{4x-2} = 0$.

723. $2^x - 2 \times (0.5)^{2x} - (0.5)^x + 1 = 0$.

724. $27 \times 2^{-3x} + 9 \times 2^x - 2^{3x} - 27 \times 2^{-x} = 8$.

725. $(2+\sqrt{3})^{x^2-2x+1} + (2-\sqrt{3})^{x^2-2x-1} = \frac{4}{2-\sqrt{3}}$.

726. $3^x \times 8^{\frac{x}{x+2}} = 6$. 727. $5^{x-2} \times 2^{\frac{3x}{x+1}} = 4$.

728. $(x^2 - x - 1)^{x^2-1} = 1$. 729. $|x|^{x^2-2x} = 1$.

730. $(x-2)^{x^2-x} = (x-2)^{12}$. 731. $(3x-4)^{2x^2+2} = (3x-4)^{5x}$.

732. $3^x + 4^x = 5^x$. 733. $8 - x \times 2^x + 2^{3-x} - x = 0$.

734. $\sqrt{x}\,(9^{\sqrt{x^2-3}} - 3^{\sqrt{x^2-3}}) = 3^{2\sqrt{x^2-3}+1} - 3^{\sqrt{x^2-3}+1} + 6\sqrt{x} - 18$.

735. $2\sqrt{x} \times 4^x + 5 \times 2^{x+1} + 2\sqrt{x} = 2^{2x+2} + 5\sqrt{x} \times 2^x + 4$.

SEC. 14. LOGARITHMIC EQUATIONS

When solving logarithmic equations, we use two basic methods: (1) replacing the equation $\log_a f(x) = \log_a g(x)$ by $f(x) = g(x)$; (2) introducing new variables. Sometimes we have to apply artificial methods.

1. Logarithmic Equations. Consider logarithmic equations of the form

$$\log_a f(x) = \log_a g(x), \tag{1}$$

where $a > 0$ and $a \neq 1$.

The solution of such equations is based on the following theorem.

Theorem 1. *The equation* $\log_a f(x) = \log_a g(x)$ *is equivalent to the mixed system*:

$$\begin{cases} f(x) = g(x) \\ f(x) > 0 \\ g(x) > 0. \end{cases} \tag{2}$$

Note that for solving Equation (1) we have not necessarily to solve System (2). We may proceed in a different way, namely, to solve the equation

$$f(x) = g(x) \tag{3}$$

and from the found solutions to choose those which satisfy the system of inequalities

$$\begin{cases} f(x) > 0 \\ g(x) > 0, \end{cases} \tag{4}$$

that is, those which belong to the domain of definition of Equation (1).

When solving logarithmic equations, we use the properties of logarithms. Consider, for instance, the equation

$$\log_a f(x) + \log_a g(x) = \log_a h(x). \tag{5}$$

It is transformed to:

$$\log_a (f(x)\ g(x)) = \log_a h(x). \tag{6}$$

But Equations (5) and (6) may be non-equivalent. Indeed, the domain of definition of the expression $\log_a f(x) + \log_a g(x)$ is given by the system of inequalities $\begin{cases} f(x) > 0 \\ g(x) > 0 \end{cases}$, whereas the domain of definition of the expression $\log_a (f(x)\ g(x))$ is specified by the inequality $f(x)\ g(x) > 0$ which is, in turn, equivalent to the collection of systems of inequalities:

$$\begin{cases} f(x) > 0 \\ g(x) > 0 \end{cases}; \qquad \begin{cases} f(x) < 0 \\ g(x) < 0. \end{cases}$$

Thus, when passing from Equation (5) to Equation (6), we can encounter an extension of the domain of definition of Equation (5) (at the expense of the solutions of the last system of inequalities), and, hence, extraneous roots may appear. Therefore, on solving Equa-

tion (6), we have to choose those of its roots which belong to the domain of definition of the original equation (5), that is, which satisfy the system of inequalities $\begin{cases} f(x) > 0 \\ g(x) > 0 \\ h(x) > 0. \end{cases}$ This check is an essential part of the solution of a logarithmic equation.

It is clear that the check may also be realized by a direct substitution of the found solutions into the original equation.

Now, consider equations of the form

$$\log_{a(x)} f(x) = \log_{a(x)} g(x). \tag{7}$$

Their solution is based on the following theorem.

Theorem 2. *Equation* (7) *is equivalent to the mixed system*:

$$\begin{cases} f(x) = g(x) \\ f(x) > 0 \\ g(x) > 0 \\ a(x) > 0 \\ a(x) \neq 1. \end{cases}$$

In other words, the roots of Equation (7) are represented by those and only those roots of the equation $f(x) = g(x)$ which simultaneously satisfy the conditions:

$$f(x) > 0, \quad g(x) > 0, \quad a(x) > 0, \quad a(x) \neq 1$$

(these conditions specify the domain of definition of Equation (7)).

Example 1. Solve the equation

$$\log_3 (x^2 - 3x - 5) = \log_3 (7 - 2x). \tag{8}$$

Solution. By Theorem 1, Equation (8) is equivalent to the following mixed system:

$$\begin{cases} x^2 - 3x - 5 = 7 - 2x \\ x^2 - 3x - 5 > 0 \\ 7 - 2x > 0. \end{cases} \tag{9}$$

Solving the equation of this system, we get: $x_1 = 4$, $x_2 = -3$. Of these two values only $x = -3$ satisfies both inequalities of System (9) (that is, the value $x = 4$ does not belong to the domain of definition of Equation (8)). Therefore, $x = -3$ is the solution of Equation (8).

Example 2. Solve the equation

$$\log (x + 4) + \log (2x + 3) = \log (1 - 2x). \tag{10}$$

Solution. We transform Equation (10) to the form

$$\log((x+4)(2x+3)) = \log(1-2x),$$

and further

$$(x+4)(2x+3) = 1-2x. \qquad (11)$$

From Equation (11) we find: $x_1 = -1$, $x_2 = -5.5$.

The domain of definition of Equation (10) is given by the system of inequalities:

$$\begin{cases} x+4>0 \\ 2x+3>0 \\ 1-2x>0. \end{cases} \qquad (12)$$

Substituting the found roots of Equation (11) into System (12), we make sure that $x_1 = -1$ satisfies this system, while $x_2 = -5.5$ does not. Thus, $x = -1$ is the only root of Equation (10).

Example 3. Solve the equation

$$\log_2(x^2-1) = \log_{\frac{1}{2}}(x-1). \qquad (13$$

Solution. First of all, let us pass in Equation (13) to logarithms with equal bases. Since $\log_a N = \log_{a^k} N^k$, Equation (13) is transformed to the following:

$$\log_2(x^2-1) = \log_{\left(\frac{1}{2}\right)^{-1}}(x-1)^{-1}.$$

Further, we have: $\log_2(x^2-1) = -\log_2(x-1)$,

$$\log_2(x^2-1) = \log_2\frac{1}{x-1}. \qquad (14)$$

Solving Equation (14), we find:

$$x_1 = 0,\ x_2 = \frac{1+\sqrt{5}}{2},\quad x_3 = \frac{1-\sqrt{5}}{2}.$$

It remains only to choose from the found values those which satisfy the system of inequalities $\begin{cases} x^2-1>0 \\ x-1>0. \end{cases}$

Solving this system, we find that $x>1$. Of the found values x_1, x_2, x_3, only $x_2 = \frac{1+\sqrt{5}}{2}$ satisfies the inequality $x>1$. Hence, $x = \frac{1+\sqrt{5}}{2}$ is the only root of Equation (13).

Example 4. Solve the equation

$$\log_{x+4}(x^2-1) = \log_{x+4}(5-x). \qquad (15)$$

Solution. By Theorem 2, this equation is equivalent to the system

$$\begin{cases} x^2-1=5-x \\ x^2-1>0 \\ 5-x>0 \\ x+4>0 \\ x+4\neq 1. \end{cases} \tag{16}$$

Solving the equation, entering System (16), we get: $x_1 = 2$, $x_2 = -3$. Of these two values only $x = 2$ satisfies the rest of the conditions of System (16). Thus, $x = 2$ is a root of Equation (15).

Example 5. Solve the equation $\log^2 x + \log x + 1 = \dfrac{7}{\log \frac{x}{10}}$.

Solution. Since $\log \frac{x}{10} = \log x - 1$, the given equation can be rewritten:

$$\log^2 x + \log x + 1 = \frac{7}{\log x - 1}.$$

Setting $u = \log x$, we get the equation

$$u^2 + u + 1 = \frac{7}{u-1},$$

whence we find: $u = 2$. From the equation $\log x = 2$ we find: $x = 100$. This is just the only root of the original equation.

Example 6. Solve the equation

$$\log^2 x^3 - \log (0.1x^{10}) = 0. \tag{17}$$

Solution. We have:

$$(\log x^3)^2 - \log x^{10} - \log 0.1 = 0,$$

$$9 \log^2 x - 10 \log |x| + 1 = 0,$$

and further

$$9 \log^2 x - 10 \log x + 1 = 0$$

(in this case $|x| = x$ since the domain of definition of Equation (17) is given by the inequality $x > 0$).

Setting $u = \log x$, we get the quadratic equation $9u^2 - 10u + 1 = 0$, whose roots are: $u_1 = 1$, $u_2 = \frac{1}{9}$. It remains to solve the collection of equations: $\log x = 1$; $\log x = \frac{1}{9}$.

From the first equation we find: $x_1 = 10$, from the second: $x_2 = \sqrt[9]{10}$. These values of x are also solutions of Equation (17).

Example 7. Solve the equation

$$\log_{0.5x} x^2 - 14 \log_{16x} x^3 + 40 \log_{4x} \sqrt{x} = 0.$$

Solution. Changing the base of all the logarithms to 2, we get:

$$\frac{\log_2 x^2}{\log_2 0.5x} - \frac{14 \log_2 x^3}{\log_2 16x} + \frac{40 \log_2 \sqrt{x}}{\log_2 4x} = 0,$$

and further

$$\frac{2 \log_2 |x|}{\log_2 x - 1} - \frac{42 \log_2 x}{\log_2 x + 4} + \frac{20 \log_2 x}{\log_2 x + 2} = 0.$$

From the given equation it follows that $x > 0$, and therefore $|x| = x$, that is, $\log_2 |x| = \log_2 x$. Setting $u = \log_2 x$, we get the equation

$$\frac{2u}{u-1} - \frac{42u}{u+4} + \frac{20u}{u+2} = 0,$$

whose roots are: $u_1 = -\frac{1}{2}$, $u_2 = 0$, $u_3 = 2$.

Now, the problem is reduced to solving the following collection of equations:

$$\log_2 x = -\frac{1}{2}; \ \log_2 x = 0; \ \log_2 x = 2.$$

From the first equation we get: $x_1 = \frac{\sqrt{2}}{2}$, from the second: $x_2 = 1$, from the third: $x_3 = 4$.

All the found values are the roots of the original equation (the reader is invited to make this sure independently).

Example 8. Solve the equation

$$\log (20 - x) = \log^3 x.$$

Solution. We fail to solve this equation using the methods considered in the previous examples. Let us find one of its roots using the trial and error method. In this case we get $x_1 = 10$. But we have no right to assert that the equation is already solved since it may have some other roots. Let us prove that there are no other roots. It is clear that the roots of the given equation should be sought for in its domain of definition, that is, in the interval (0, 20). In this interval the function $y = \log (20 - x)$ decreases, while the function $y = \log^3 x$ increases. But then the equation has only one root (see Item 3 of Sec. 12). Thus, $x = 10$ is the only root of the given equation.

2. Exponential-logarithmic Equations. Here, we shall consider equations which may be regarded as both exponential and logarithmic.

Example 9. Solve the equation

$$x^{1 - \log x} = 0.01. \tag{18}$$

Solution. The domain of definition of the equation is $x > 0$. In this domain the expressions contained on both sides of Equa-

tion (18) take on only positive values, therefore taking the decimal logarithms of both sides of the equation, we get the equation

$$\log x^{1-\log x} = \log 0.01$$

which is equivalent to Equation (18). Further, we have:

$$(1 - \log x) \log x = -2.$$

Setting $u = \log x$, we get the equation $(1 - u)\, u = -2$, whence $u_1 = -1$, $u_2 = 2$. It remains to solve the following collection of equations: $\log x = -1$; $\log x = 2$.

From this collection we get: $x_1 = 0.1$, $x_2 = 100$. These are roots of Equation (18).

Example 10. Solve the equation

$$\log_x (3x^{\log_5 x} + 4) = 2 \log_5 x. \tag{19}$$

Solution. Using the definition of logarithm, we transform Equation (19) to:

$$x^{2 \log_5 x} = 3x^{\log_5 x} + 4.$$

Setting $u = x^{\log_5 x}$, we get the equation $u^2 - 3u - 4 = 0$, whose roots are: $u_1 = -1$, $u_2 = 4$.

Now, the problem is reduced to solving the following collection of equations: $x^{\log_5 x} = -1$; $x^{\log_5 x} = 4$.

Since $x^{\log_5 x} > 0$, and $-1 < 0$, the first equation of this collection has no solution. Taking the logarithms to the base 5 of both sides of the second equation, we get:

$$\log_5^2 x = \log_5 4, \quad \text{i.e.} \quad \log_5 x = \pm \sqrt{\log_5 4},$$

whence we find: $x_{1,2} = 5^{\pm\sqrt{\log_5 4}}$. These are roots of Equation (19).

Example 11. Solve the equation

$$\log_5 (5^{\frac{1}{x}} + 125) = \log_5 6 + 1 + \frac{1}{2x}. \tag{20}$$

Solution. Let us first consider the given equation as a logarithmic one. Since $1 + \frac{1}{2x} = \log_5 5^{1+\frac{1}{2x}}$, we write Equation (20) in the form:

$$\log_5 (5^{\frac{1}{x}} + 125) = \log_5 6 + \log_5 5^{1+\frac{1}{2x}}.$$

Further, we have:

$$\log_5 (5^{\frac{1}{x}} + 125) = \log_5 (6 \times 5 \times 5^{\frac{1}{2x}}), \quad 5^{\frac{1}{x}} + 125 = 30 \times 5^{\frac{1}{2x}}.$$

We have obtained an exponential equation which can be solved by introducing a new variable. Setting $u = 5^{\frac{1}{2x}}$, we get the equation $u^2 - 30u + 125 = 0$, whose roots are: $u_1 = 5$, $u_2 = 25$.

Now, the problem is reduced to solving the collection of two equations

$$5^{\frac{1}{2x}} = 5; \; 5^{\frac{1}{2x}} = 25.$$

From the first equation we get: $\frac{1}{2x} = 1$, whence $x_1 = \frac{1}{2}$.

From the second equation we get: $\frac{1}{2x} = 2$, whence $x_2 = \frac{1}{4}$.

Thus, Equation (20) has two roots: $x_1 = \frac{1}{2}$ and $x_2 = \frac{1}{4}$.

EXERCISES

In Problems 736 through 805, solve the indicated equations:

736. $\log_4 \frac{2}{x-1} = \log_4 (4 - x)$. **737.** $\log_3 ((x - 1)(2x - 1)) = 0$.

738. $\log_{\sqrt{2}} \frac{x^2 - 4x + 3}{4} = -2$. **739.** $\log (x + 1.5) = -\log x$.

740. $\log (4.5 - x) = \log 4.5 - \log x$.

741. $\log \sqrt{5x - 3} + \log \sqrt{x + 1} = 2 + \log 0.018$.

742. $\log \sqrt{x - 5} + \log \sqrt{2x - 3} + 1 = \log 30$.

743. $\log (x^3 + 27) - 0.5 \log (x^2 + 6x + 9) = 3 \log \sqrt[3]{7}$.

744. $\log 5 + \log (x + 10) = 1 - \log (2x - 1) + \log (21x - 20)$.

745. $\log_5 (3x - 11) + \log_5 (x - 27) = 3 + \log_5 8$.

746. $\frac{1 - \log x}{x} = \frac{\log^2 14 - \log^2 4}{\log 3.5^x}$.

747. $\log_2 (x + 1)^2 + \log_2 \sqrt{x^2 + 2x + 1} = 6$.

748. $\frac{\log (35 - x^3)}{\log (5 - x)} = 3$. **749.** $\frac{\log 2 + \log (4 - 5x - 6x^2)}{\log \sqrt[3]{2x - 1}} = 3$.

750. $\log_{\frac{1}{5}} \log_5 \sqrt{5x} = 0$. **751.** $\log_{\frac{1}{2}} \log_8 \frac{x^2 - 2x}{x - 3} = 0$

752. $\log_4 \log_2 \log_3 (2x - 1) = \frac{1}{2}$.

753. $\log_{\frac{1}{2}} \sqrt{1 + x} + 3 \log_{\frac{1}{4}} (1 - x) = \log_{\frac{1}{16}} (1 - x^2)^2 + 2$.

754. $(1 - \log 2) \log_5 x = \log 3 - \log (x - 2)$.

755. $\log_{x^2} (x + 2) = 1$. **756.** $\log_{x-2} (2x - 9) = \log_{x-2} (23 - 6x)$.

757. $\log_{5x-2} 2 + 2 \log_{5x-2} x = \log_{5x-2} (x + 1)$.

758. $\log_4 (x + 12) \cdot \log_x 2 = 1$. **759.** $x^2 \cdot \log_x 27 \cdot \log_9 x = x + 4$.

760. $1+\log_x \frac{4-x}{10}=(\log x^2-1)\log_x 10.$

761. $1+2\log_x 2\cdot\log_4(10-x)=\frac{2}{\log_4 x}.$ 762. $\log_{x+\frac{1}{8}} 2=\log_x 4.$

763. $\log_3(-x^2-8x-14)\log_{x^2+4x+4} 9=1.$ 764. $0.1\log^4 x-\log^2 x+0.9=0.$

765. $\frac{1}{5-4\log(x+1)}+\frac{5}{1+4\log(x+1)}=2.$ 766. $4-\log x=3\sqrt{\log x}.$

767. $\log^2(100x)+\log^2(10x)=14\log x+15.$

768. $\frac{1-\log^2 x^2}{\log x-2\log^2 x}=\log x^4+5.$ 769. $\log_x 5\sqrt{5}-\frac{5}{4}=\log_x^2\sqrt{5}.$

770. $\log(\log x)+\log(\log x^3-2)=0.$

771. $\log_x 2+\log_2 x=2.5.$ 772. $\log^2_{\frac{1}{2}} 4x+\log_2\frac{x^2}{8}=8.$

773. $\log(x^2-8)\cdot\log(2-x)=\frac{\log_5(x^2-8)}{\log_5(2-x)}.$

774. $\log_2\log_3(x^2-16)-\log_{\frac{1}{2}}\log_{\frac{1}{3}}\frac{1}{x^2-16}=2.$

775. $3\log_{16}(\sqrt{x^2+1}+x)+\log_2(\sqrt{x^2+1}-x)=\log_{16}(4x+1)-0.5.$

776. $2\log_x 3+\log_{3x} 3+3\log_{9x} 3=0.$ 777. $\log_{x+1}\left(x-\frac{1}{2}\right)=\log_{x-\frac{1}{2}}(x+1).$

778. $\log_{3x+7}(5x+3)+\log_{5x+3}(3x+7)=2.$ 779. $(0.4)^{\log^2 x+1}=(6.25)^{2-\log x^3}.$

780. $(1.25)^{1-\log_2^2 x}=(0.64)^{2\log_2 2x}.$ 781. $x^{\log x}=1000x^2.$

782. $\sqrt{x^{\log\sqrt{x}}}=10.$ 783. $x^{\log_{\sqrt{x}} 2x}=4.$

784. $x^{\frac{\log x+7}{4}}=10^{\log x+1}.$ 785. $(\sqrt{x})^{\log_5 x-1}=5.$

786. $x^{\log_3 x+1}=9x^2.$ 787. $(\sqrt{x})^{\log_{x^2}(x^2-1)}=5.$

788. $\log_x(2x^{x-2}-1)+4=2x.$ 789. $15^{\log_5 3}x^{\log_5 9x+1}=1.$

790. $16^{\log_x 2}=8x.$ 791. $x^{\log_2 x^3-\log_2^2 x+3}-\frac{1}{x}=0.$

792. $5^{\log x}-3^{\log x-1}=3^{\log x+1}-5^{\log x-1}.$

793. $2x^{\log x}+3x^{-\log x}=5.$ 794. $x^{(\log_3 x)^2-3\log_3 x}=3^{8-3\log_{2\sqrt{2}} 4}.$

795. $x^{\log_2^2 x^2-\log_2 2x-2}+(x+2)^{\log_{(x+2)^2} 4}=3.$

796. $7x^{\frac{1}{\log_2^2 x^3}+\log_x 2}=5+(x+7)^{\frac{2}{\log_{\sqrt{2}}(x+7)}}.$

797. $\log(3^x-2^{4-x})=2+0.25\log 16-0.5x\log 4.$

798. $\log_2(9-2^x)=25^{\log_5\sqrt{3-x}}.$

799. $3\log 2+\log(2^{\sqrt{x-1}-1}-1)=\log(0.4\sqrt{2^{\sqrt{x-1}}+4})+1.$

800. $|x-1|^{\log^2 x-\log x^2}=|x-1|^3.$

801. $4^{\log_3(1-x)}=(2x^2+2x+5)^{\log_3 2}.$ **802.** $x^{\log_2 \frac{x}{98}}\times 14^{\log_2 7}=1.$

803. $3x+(3-x)\log_3 2=\log_3\left(9\times\left(\frac{8}{3}\right)^x+2\times 6^x\right)+1.$

804. $x^2\log_6(5x^2-2x-3)-x\log_{\frac{1}{6}}(5x^2-2x-3)=x^2+x.$

805. $x^2\log_2\frac{3+x}{10}-x^2\log_{\frac{1}{2}}(2+3x)=x^2-4+2\log_{\sqrt{2}}\frac{3x^2+11x+6}{10}.$

SEC. 15. SYSTEMS OF EXPONENTIAL AND LOGARITHMIC EQUATIONS

When solving systems of exponential and logarithmic equations, we use the same techniques as were applied in solving systems of algebraic equations. However, we should like to underline that in many cases, prior to applying this or that method of solution, each equation of a system should be simplified.

Example 1. Solve the system of equations

$$\begin{cases} 25^{2x}+25^{2y}=20 \\ 25^{x+y}=5\sqrt{5}. \end{cases} \tag{1}$$

Solution. Setting $u=25^x$, $v=25^y$, we get the system of equations $\begin{cases} u^2+v^2=30 \\ uv=5\sqrt{5}, \end{cases}$ which has four solutions:

$$\begin{cases} u_1=5 \\ v_1=\sqrt{5} \end{cases};\quad \begin{cases} u_2=\sqrt{5} \\ v_2=5 \end{cases};\quad \begin{cases} u_3=-5 \\ v_3=-\sqrt{5} \end{cases};\quad \begin{cases} u_4=-\sqrt{5} \\ v_4=-5. \end{cases}$$

But $u=25^x$, $v=25^y$, hence, $u>0$, $v>0$, that is, of the four found solutions we have to take only the first two.

Thus, the problem is reduced to solving the following collection of systems of equations:

$$\begin{cases} 25^x=5 \\ 25^y=\sqrt{5} \end{cases};\quad \begin{cases} 25^x=\sqrt{5} \\ 25^y=5. \end{cases}$$

From the first system we find: $x_1=\frac{1}{2}$, $y_1=\frac{1}{4}$, from the second: $x_2=\frac{1}{4}$, $y_2=\frac{1}{2}$.

Thus, System (1) has two solutions: $\left(\frac{1}{2}, \frac{1}{4}\right)$, $\left(\frac{1}{4}, \frac{1}{2}\right)$.

Example 2. Solve the system

$$\begin{cases} 2^x 3^y = 12 \\ 2^y 3^x = 18. \end{cases} \tag{2}$$

Solution. Multiplying termwise the equations of System (2), we get the equation

$$2^{x+y} 3^{x+y} = 216 \text{ or } 6^{x+y} = 6^3,$$

whence $x + y = 3$.

Dividing termwise the first equation of System (2) by the second, we get the equation

$$2^{x-y} 3^{y-x} = \frac{2}{3} \text{ or } \left(\frac{2}{3}\right)^{x-y} = \frac{2}{3},$$

whence $x - y = 1$.

Further, from the system of equations $\begin{cases} x + y = 3 \\ x - y = 1 \end{cases}$ we find: $x = 2$, $y = 1$.

Thus, the pair (2, 1) is the solution of System (2).

Example 3. Solve the system

$$\begin{cases} x^{y-2} = 4 \\ x^{2y-3} = 64. \end{cases} \tag{3}$$

Solution. Taking the logarithms to the base 2 of both sides of each of the equations of System (3), we get the following system of equations:

$$\begin{cases} \log_2 x^{y-2} = 2 \\ \log_2 x^{2y-3} = 6 \end{cases} \text{ or } \begin{cases} (y-2) \log_2 x = 2 \\ (2y-3) \log_2 x = 6. \end{cases}$$

It is clear that $x \neq 1$, that is, $\log_2 x \neq 0$. Therefore, dividing the first equation of this system by the second, we get: $\frac{y-2}{2y-3} = \frac{1}{3}$, whence $y = 3$. Substituting this value of y into the equation $(y - 2) \log_2 x = 2$, we find: $\log_2 x = 2$, that is, $x = 4$.

Thus, the pair (4, 3) is the solution of System (3).

Example 4. Solve the system of equations

$$\begin{cases} x^{\log_y x} \cdot y = x^{\frac{5}{2}} \\ \log_4 y \cdot \log_y (y - 3x) = 1. \end{cases} \tag{4}$$

Solution. Let us reduce the first equation of System (4) to a simpler form. For this purpose, we take the logarithm to the base y of both sides of the equation:

$$\log_y(x^{\log_y x}\cdot y)=\log_y x^{\frac{5}{2}},$$

and further

$$\log_y x^{\log_y x}+\log_y y=\frac{5}{2}\log_1 x,\quad \log_y^2+1=\frac{5}{2}\log_y x.$$

Setting $u=\log_y x$, we get the quadratic (with respect to u) equation $u^2-\frac{2}{5}u+1=0$, whose roots are: $u_1=2$, $u_2=\frac{1}{2}$. Hence, either $\log_y x=2$ and therefore $x=y^2$, or $\log_y x=\frac{1}{2}$, then $x=\sqrt{y}$, i.e. $y=x^2$. Thus, the first equation of System (4) is reduced to the collection of two simpler equations:

$$x=y^2;\quad y=x^2.$$

Let us now reduce the second equation of System (4) to a simpler form. To this end, we change the base of the logarithm to 4:

$$\log_4 y\cdot\frac{\log_4(y-3x)}{\log_4 y}=1,$$

and, further, $\log_4(y-3x)=1$, whence $y-3x=4$.

It now remains to solve the collection of two simple systems of equations:

$$\begin{cases}x=y^2\\ y-3x=4\end{cases};\quad \begin{cases}y=x^2\\ y-3x=4.\end{cases}$$

The first system has no solution, the second has two solutions: (4, 16), (−1, 1).

Check. The solutions of System (4) must satisfy the following conditions:

$$\begin{cases}x>0\\ y>0\\ y-3x>0\\ y\neq 1.\end{cases}$$

The pair (4, 16) satisfies this system, whereas the pair (−1, 1) does not. Hence, (4, 16) is the only solution of System (4).

EXERCISES

In Problems 806 through 834, solve the given systems of equations:

806. $\begin{cases}\left(\frac{3}{2}\right)^{x-y}-\left(\frac{2}{3}\right)^{x-y}=\frac{65}{36}\\ xy-x+y=118.\end{cases}$ 807. $\begin{cases}2^x+2^y=12\\ x+y=5.\end{cases}$

808. $\begin{cases}64^{2x}+64^{2y}=12\\ 64^{x+y}=4\sqrt{2}.\end{cases}$

809. $\begin{cases}8^x=10y\\ 2^x=5y.\end{cases}$ 810. $\begin{cases}2^x\times 9^y=648\\ 3^x\times 4^y=432.\end{cases}$

811. $\begin{cases}3^x-2^{2y}=77\\ 3^{\frac{x}{2}}-2^y=7.\end{cases}$ 812. $\begin{cases}x^{y+1}=27\\ x^{2y-5}=\frac{1}{3}.\end{cases}$

813. $\begin{cases}x^{x+y}=y^{12}\\ y^{x+y}=x^3.\end{cases}$ 814. $\begin{cases}x^{\sqrt{y}}=y\\ y^{\sqrt{y}}=x^4.\end{cases}$

815. $\begin{cases}\log x+\log y=\log 2\\ x^2+y^2=5.\end{cases}$ 816. $\begin{cases}\log_y x-\log_x y=\frac{8}{3}\\ xy=16.\end{cases}$

817. $\begin{cases}\log(x^2+y^2)-1=\log 13\\ \log(x+y)-\log(x-y)=3\log 2.\end{cases}$ 818. $\begin{cases}5(\log_y x+\log_x y)=26\\ xy=64.\end{cases}$

819. $\begin{cases}2^x\times 4^y=32\\ \log(x-y)^2=2\log 2.\end{cases}$ 820. $\begin{cases}10^{2-\log(x-y)}=25\\ \log(x-y)+\log(x+y)=1+2\log 2.\end{cases}$

821. $\begin{cases}2^{\frac{x-y}{2}}-(\sqrt[4]{2})^{x-y}=12\\ 3^{\log(2y-x)}=1.\end{cases}$ 822. $\begin{cases}3^x\times 2^y=576\\ \log_{\sqrt{2}}(y-x)=4.\end{cases}$

823. $\begin{cases}\log_5 x+3^{\log_3 y}=7\\ x^y=5^{12}.\end{cases}$ 824. $\begin{cases}3(2\log_{y^2} x-\log_{\frac{1}{x}} y)=10\\ xy=81.\end{cases}$

825. $\begin{cases}\log_{0.5}(y-x)+\log_2\frac{1}{y}=-2\\ x^2+y^2=25.\end{cases}$

826. $\begin{cases}(x+y)\,3^{y-x}=\frac{5}{27}\\ 3\log_5(x+y)=x-y.\end{cases}$ 827. $\begin{cases}x^y=y^x\\ x^x=y^{9y}\ (x>0,\ y>0).\end{cases}$

828. $\begin{cases}20x^{\log_3 y}+7y^{\log_3 x}=81\sqrt[3]{3}\\ \log_9 x^2+\log_{27} y^3=\frac{8}{3}.\end{cases}$

829. $\begin{cases}\log_4 xy+3\dfrac{\log_4 x}{\log_4 y}=0\\ \log_4\dfrac{x}{y}-\log_4 x\cdot\log_4 y=0.\end{cases}$

830. $\begin{cases} \log_2(x+y)+2\log_3(x-y)=5 \\ 2^x-5\times 2^{0.5(x+y-1)}+2^{y+1}=0. \end{cases}$

831. $\begin{cases} \log_2(10-2^y)=4-y \\ \log_2 \dfrac{x+3y-1}{3y-x}=\log_2(x-1)-\log_2(3-x). \end{cases}$

832. $\begin{cases} \log x \log(x+y)=\log y \log(x-y) \\ \log y \log(x+y)=\log x \log(x-y). \end{cases}$

833. $\begin{cases} 4^{x+y}=27+9^{x-y} \\ 8^{x+y}-21\times 2^{x+y}=27^{x-y}+7\times 3^{x-y+1}. \end{cases}$

834. $\begin{cases} x\times 2^{x+1}-2\times 2^y=-3y\times 4^{x+y} \\ 2x\times 2^{2x+y}+3y\times 8^{x+y}=1. \end{cases}$

SEC. 16. RATIONAL INEQUALITIES

1. Basic Concepts. The *domain of definition of the inequality* $f(x) > g(x)$ is defined as the set of all x's where both functions $f(x)$ and $g(x)$ are defined. In other words, the domain of definition of the inequality $f(x) > g(x)$ is the intersection of the domains of definition of $f(x)$ and $g(x)$.

A *particular solution of the inequality* $f(x) > g(x)$ is defined as a value of the variable x satisfying this inequality (that is, any value of x for which the statement "the value of the function $f(x)$ is greater than the value of the function $g(x)$" is true). The *solution of an inequality* is understood as the set of its particular solutions.

Two inequalities in one variable x are said to be *equivalent* if their solutions coincide (in particular, if both inequalities have no solutions). If each particular solution of the inequality $f_1(x) > g_1(x)$ is a particular solution of the inequality $f_2(x) > g_2(x)$ obtained as the result of transformations of the inequality $f_1(x) > g_1(x)$ (that is, if the solution of the first inequality enters the solution of the second inequality), then the inequality $f_2(x) > g_2(x)$ is said to be a *consequence of the inequality* $f_1(x) > g_1(x)$. The following theorems dwell on transformations leading to equivalent inequalities.

Theorem 1. *If to both sides of an inequality the same function* $\varphi(x)$ *is added which is defined for all x's from the domain of definition of the original inequality and the sense of the inequality is left unchanged, then the obtained inequality is equivalent to the given inequality.*

Thus, the inequalities

$$f(x) > g(x) \text{ and } f(x)+\varphi(x) > g(x)+\varphi(x)$$

are equivalent if $\varphi(x)$ satisfies the conditions of the theorem.

Corollary. *The inequalities*

$$f(x)+\varphi(x) > g(x) \textit{ and } f(x) > g(x)-\varphi(x)$$

are equivalent.

Theorem 2. *If both sides of an inequality are multiplied (or divided) by the same function $\varphi(x)$, which for all x's from the domain of definition of the original inequality takes on only positive values, and the sense of the inequality is left unchanged, then the obtained inequality is equivalent to the given.*

Thus, if $\varphi(x) > 0$, then the inequalities

$$f(x) > g(x) \text{ and } f(x)\varphi(x) > g(x)\varphi(x)$$

$\left(\text{or } \frac{f(x)}{\varphi(x)} > \frac{g(x)}{\varphi(x)}\right)$ are equivalent.

Corollary. *If both sides of an inequality are multiplied (or divided) by the same positive number and the sense of the inequality is left unchanged, then the obtained inequality is equivalent to the given.*

Theorem 3. *If both sides of an inequality are multiplied (or divided) by the same function $\varphi(x)$, which for all x's from the domain of definition of the original inequality takes on only negative values, and the sense of the inequality is reversed, then the obtained inequality is equivalent to the given.*

Thus, if $\varphi(x) < 0$, then the inequalities

$$f(x) > g(x) \text{ and } f(x)\varphi(x) < g(x)\varphi(x)$$

$\left(\text{or } \frac{f(x)}{\varphi(x)} < \frac{g(x)}{\varphi(x)}\right)$ are equivalent.

Corollary. *If both sides of an inequality are multiplied (or divided) by the same negative number and the sense of the inequality is reversed, then the obtained inequality is equivalent to the given inequality.*

Theorem 4. *Let there be given an inequality $f(x) > g(x)$, where $f(x) \geqslant 0$ and $g(x) \geqslant 0$ for all x's from the domain of definition of the inequality. If both sides of the inequality are raised to the same natural power n and the sense of the inequality is left unchanged, then the following inequality is obtained:*

$$(f(x))^n > (g(x))^n,$$

which is equivalent to the given inequality.

Remark. In Sec. 7 we already marked that identical transformations might cause a change in the domain of definition of a function. For instance, collection of like terms and reduction of a fraction may result in an extension of the domain of definition. When solving an inequality we use identical transformations which may yield a non-equivalent inequality. Consider, as an example, the inequality

$$\sqrt{x} + x - 1 > \sqrt{x} - 5. \qquad (1)$$

Adding the same function $\varphi(x) = -\sqrt{x}$ to both sides of the inequality, we get the inequality

$$\sqrt{x} + x - 1 - \sqrt{x} > \sqrt{x} - 5 - \sqrt{x}, \qquad (2)$$

equivalent (by Theorem 1) to Inequality (1). Further we have:

$$x - 1 > -5, \tag{3}$$

whence $x > -4$. But Inequality (1) has the solution $x \geqslant 0$, that is, Inequalities (1) and (3) are not equivalent (Inequality (3) is a consequence of Inequality (1)). The thing is that the inequality $x - 1 > -5$ has a wider domain of definition as compared with Inequality (1); this extension is due to collecting like terms in Inequality (2). Therefore, after carrying out identical transformations resulting in an extension of the domain of definition of the inequality, we have to choose those of the found solutions which belong to the domain of definition of the original inequality.

2. **Rational Inequalities.** Consider the function

$$f(x) = \frac{(x-a_1)^{n_1}(x-a_2)^{n_2}\dots(x-a_k)^{n_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}\dots(x-b_p)^{m_p}}, \tag{4}$$

where $n_1, n_2, \dots, n_k, m_1, m_2, \dots, m_p$ are natural numbers, and the numbers $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_p$ are any numbers such that $a_i \neq b_j$, where $i = 1, 2, \dots, k$, $j = 1, 2, \dots, p$. An inequality

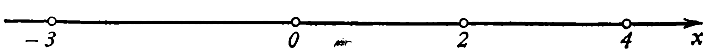

Fig. 6

of the form $f(x) > 0$, where $f(x)$ is defined by (4), is called a *rational inequality*. At points $x = a_i$ the function $f(x)$ vanishes (these points are called *function zeros*). The points $x = b_j$ are the points of discontinuity of the function $f(x)$. Marking all function zeros and points of discontinuity on the number line, we separate the latter into $k + p + 1$ intervals. As is known from the course of mathematical analysis, within each of these intervals the function $f(x)$ is continuous and preserves the sign. To determine this sign, it suffices to find the sign of the function at any point from the interval we are interested in.

Example 1. Solve the inequality $\frac{x^2(x-2)^3(x+3)}{(x-4)^7} > 0$.

Solution. The function $f(x) = \frac{x^2(x-2)^3(x+3)}{(x-4)^7}$ vanishes at the points $x_1 = 0$, $x_2 = 2$, $x_3 = -3$ and has a discontinuity at point $x_4 = 4$. These four points divide the number line into five intervals (Fig. 6): $(-\infty, -3)$, $(-3, 0)$, $(0, 2)$, $(2, 4)$, $(4, \infty)$. Let us determine the sign of the function $f(x)$ within each of these intervals.

We take the point $x = -4$ in the interval $(-\infty, -3)$. We have: $f(-4) < 0$, hence, $f(x) < 0$ in $(-\infty, -3)$.

We take the point $x = -1$ in the interval $(-3, 0)$. We have: $f(-1) > 0$, hence, $f(x) > 0$ in $(-3, 0)$.

We take the point $x = 1$ in the interval $(0, 2)$. We have: $f(1) > 0$, hence, $f(x) > 0$ in $(0, 2)$.

We take the point $x = 3$ in the interval $(2, 4)$. We have: $f(3) < 0$, hence, $f(x) < 0$ in $(2, 4)$.

We take the point $x = 5$ in the interval $(4, \infty)$. We have: $f(5) > 0$, hence, $f(x) > 0$ in $(4, \infty)$.

We solve the inequality $f(x) > 0$. From the above reasoning it is clear that the inequality is fulfilled within the intervals $(-3, 0)$, $(0, 2)$ and $(4, \infty)$. The union of these intervals just presents the solution of the given inequality.

The answer may be written in two ways:

(1) $(-3, 0) \cup (0, 2) \cup (4, \infty)$;

(2) $-3 < x < 0$; $0 < x < 2$; $4 < x < \infty$.

In practice, for solving the inequality $f(x) > 0$ (and also $<$, $\geqslant$, $\leqslant$), where $f(x)$ is a function of the form (4), we apply the so-called method of intervals—a geometric method of solution based on the three obvious statements:

(1) If c is the greatest of the numbers a_i, b_j, then the function $f(x)$ is positive in the interval (c, ∞).

(2) If a_i (or b_j) is such a point that the exponent n_i of the function $(x - a_i)^{n_i}$ [or $(x - b_j)^{n_j}$] is an odd number, then on the right and on the left of a_i (or of b_j), i.e. in adjacent intervals the function has

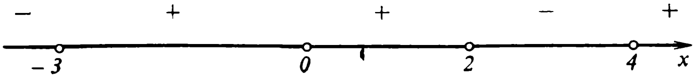

Fig. 7

unlike signs and point a_i (or b_j) is called *simple*. The above statement means that, when passing through a simple point, the function $f(x)$ changes sign.

(3) If a_i (or b_j) is a point such that the exponent n_i of the function $(x - a_i)^{n_i}$ [or $(x - b_j)^{n_j}$] is an even number, then on the right and on the left of a_i (or of b_j), i.e. in adjacent intervals the function $f(x)$ has like signs and point a_i (or b_j) is called a *double point*. The above statement means that, when passing through a double point, the function does not change sign.

Thus, in Example 1, the points $x = 2$, $x = -3$, $x = 4$ are simple, while $x = 0$ is a double point. The signs of the function $f(x)$ in the relevant intervals are shown in Fig. 7.

Hence, $f(x) > 0$ in the intervals $(-3, 0)$, $(0, 2)$ and $(4, \infty)$. The same was obtained above when solving Example 1.

The *method of intervals* based on the three statements formulated above is used for solving inequalities of the form

$$\frac{(x-a_1)^{n_1}(x-a_2)^{n_2}\dots(x-a_k)^{n_k}}{(x-b_1)^{m_1}(x-b_2)^{m_2}\dots(x-b_p)^{m_p}} > 0 \ (< 0). \qquad (5)$$

It consists in the following:

(1) All zeros and points of discontinuity of the function $f(x)$ contained on the left-hand side of Inequality (5) are marked on the number line with uninked (white) circles.

(2) From right to left, beginning above the number line, a wavy curve is drawn which passes through all the marked points so that,

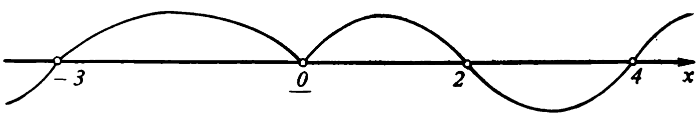

Fig. 8

when passing through a simple point, the curve intersects the number line, and, when passing through a double point, the curve remains located on one side of the number line.

(3) The appropriate intervals are chosen in accordance with the sign of Inequality (5) (the function $f(x)$ is positive whenever the curve is situated above the number line; it is negative if the curve is found below the number line); their union just represents the solution of Inequality (5).

For convenience, a number corresponding to a double point will be underlined, and the wavy curve will be called the *curve of signs.* Figure 8 represents the curve of signs for the inequality from Example 1.

Let us also note that in the non-strict inequalities $f(x) \geqslant 0$ or $f(x) \leqslant 0$, where $f(x)$ is a function of the form (4), the zeros of the function are marked with inked (black) circles in the figure and are included in the answer. Points of discontinuity are always represented by uninked circles and are not included in the answer.

Example 2. Solve the inequality $\frac{(x+5)(x-\sqrt{3})(x+\sqrt{2})}{(2x-3)(4x+5)} < 0.$

Solution. Let us transform the inequality to

$$\frac{(x+5)(x-\sqrt{3})(x+\sqrt{2})}{2\left(x-\frac{3}{2}\right)\times 4\left(x+\frac{5}{4}\right)} < 0.$$

A change in sign of the function $f(x) = \frac{(x+5)(x-\sqrt{3})(x+\sqrt{2})}{\left(x-\frac{3}{2}\right)\left(x+\frac{5}{4}\right)}$

is illustrated by a curve of signs; here all zeros and points of discontinuity are simple points (Fig. 9). The values of x for

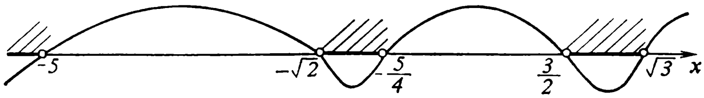

Fig. 9

which $f(x) < 0$ (they are hatched in figure) lie in the intervals $(\infty, -5)$, $\left(-\sqrt{2}, -\frac{5}{4}\right)$ and $\left(\frac{3}{2}, \sqrt{3}\right)$

The union of these intervals represents the solution of the given inequality.

Example 3. Solve the inequality $2x^3 - 5x^2 + 2x \leqslant 0$.

Solution. We have: $2x(x-2)\left(x-\frac{1}{2}\right) \leqslant 0$, and further

$$x(x-2)\left(x-\frac{1}{2}\right) \leqslant 0.$$

We now draw the curve of signs (Fig. 10). Since the given inequality is non-strict, it is also satisfied by those values of x for which

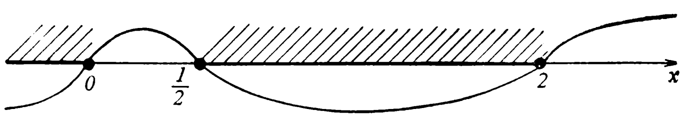

Fig. 10

the left-hand member of the inequality vanishes. These points are marked in Fig. 10 with inked circles. Thus, the given inequality has the following solution: $(-\infty, 0] \cup \left[\frac{1}{2}, 2\right]$.

Example 4. Solve the inequality $\dfrac{x^2-3x-18}{13x-x^2-42} \geqslant 0$.

Solution. Multiplying both sides of the inequality by -1, we get:

$$\frac{x^2-3x-18}{x^2-13x+42} \leqslant 0 \text{ or } \frac{(x-6)(x+3)}{(x-6)(x-7)} \leqslant 0.$$

Reducing the fraction on the left side of the inequality by $x-6$, we get: $\dfrac{x+3}{x-7} \leqslant 0$. With the aid of the curve of signs (Fig. 11 (a)) we find the interval $[-3, 7)$.

Excluding from the found set the value $x = 6$ as not belonging to the domain of definition of the original inequality (Fig. 11 (*b*)), we get the following solution: $[-3, 6) \cup (6, 7)$.

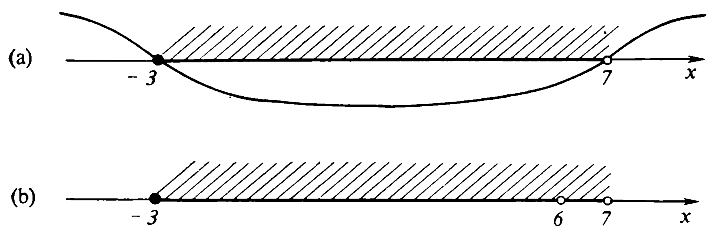

Fig. 11

Example 5. Solve the inequality $\frac{(x-3)(x+2)}{x^2-1} < 1$.

Solution. We have: $\frac{x^2-x-6}{x^2-1} - 1 < 0$, and further $\frac{x+5}{(x-1)(x+1)} > 0$. We draw the curve of signs for the function $f(x) = \frac{x+5}{(x-1)(x+1)}$ (Fig. 12) and, with the aid of this curve find the solution of this inequality: $(-5, -1) \cup (1, \infty)$.

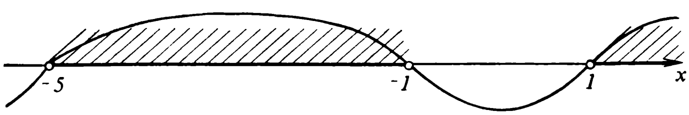

Fig. 12

Example 6. Solve the inequality $\frac{(x-1)^3 (x+2)^4 (x-3)^5 (x+6)}{x^2 (x-7)^3} \leqslant 0$.

Solution. We mark on the number line the zeros of the function: 1, -2, 3, and -6 (with shaded circles) and the points of discontinuity: 0 and 7 (with open circles), isolate the double points: -2 and 0, and draw the curve of signs (Fig. 13). We write the answer

$$-6 \leqslant x \leqslant -2; \quad -2 \leqslant x < 0; \quad 0 < x \leqslant 1; \quad 3 \leqslant x < 7$$

or more briefly $[-6, 0) \cup (0, 1] \cup [3, 7)$.

Example 7. Solve the inequality $\frac{3x+4}{x^2-3x+5} < 0$.

Solution. The discriminant D of the denominator is equal to $9 - 20 < 0$. But, as is known, if the discriminant of the quadratic trinomial $ax^2 + bx + c$, where $a > 0$, is negative, then the inequality $ax^2 + bx + c > 0$ is fulfilled for all x's.

Thus, the denominator of the left-hand member of the given inequality is positive for any values of x, therefore, multiplying both sides of the inequality by $x^2 - 3x + 5$ and retaining the sense of

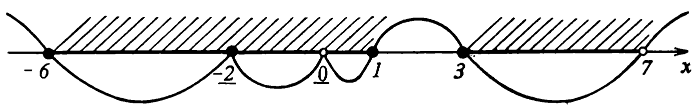

Fig. 13

the inequality, we get the inequality $3x + 4 < 0$ equivalent to the given. Its solution, and, consequently, the solution of the original inequality is represented by the number interval $\left(-\infty, -\frac{4}{3}\right)$.

3. Systems and Collections of Inequalities in One Variable. Several inequalities in one variable form a *system* of inequalities if a problem is posed to find all those values of the variable which *simultaneously satisfy each of the given inequalities.*

Several inequalities in one variable form a *collection* of inequalities if a problem is posed to find all those values of the variable each of which satisfies *at least one of the given inequalities.*

Hence it follows that the solution of a system of inequalities is an *intersection* of solutions of the inequalities forming the system; the solution of a collection of inequalities is a *union* of solutions of the inequalities entering the collection (here, as above, the solution is understood as the general solution, that is, the set of all particular solutions).

The inequalities forming a system are united by a brace. Sometimes, a system may be written in line. For instance, the system $\begin{cases} 2x+3<x-4 \\ 2x+3>3x-1 \end{cases}$ may be written in the following way: $3x - 1 < 2x + 3 < x - 4$.

From the definition of a system of inequalities it follows that if the inequality $f(x) > g(x)$ is a consequence of the inequalities $f_1(x) > g_1(x)$ and $f_2(x) > g_2(x)$ (or a consequence of only one of these inequalities), then the system of inequalities

$$\begin{cases} f_1(x) > g_1(x) \\ f_2(x) > g_2(x) \end{cases}$$

is equivalent to the following system:

$$\begin{cases} f_1(x) > g_1(x) \\ f_2(x) > g_2(x) \\ f(x) > g(x). \end{cases}$$

In other words, if a consequent inequality is appended to a given system of inequalities or, vice versa, a consequent inequality is rejected from a given system of inequalities, then the obtained system of inequalities is equivalent to the given system. Thus, the following two systems of inequalities are equivalent:

$$\begin{cases} x^2-5x>3 \\ x^2-5x>7 \\ \dfrac{2x-1}{x+2}<1 \end{cases} \quad \text{and} \quad \begin{cases} x^2-5x>7 \\ \dfrac{2x-1}{x+2}<1 \end{cases}$$

(here, the inequality $x^2 - 5x > 3$, which is a consequence of the inequality $x^2 - 5x > 7$, has been rejected).

The inequalities forming a collection are united by a bracket. A collection of inequalities may also be written in line; in such a case the inequalities are separated by a semicolon.

A non-strict inequality is equivalent to a collection consisting of a strict inequality and an equation. For instance, the inequality $f(x) \geqslant g(x)$ is equivalent to the collection

$$\left[\begin{array}{l} f(x) > g(x) \\ f(x) = g(x). \end{array}\right.$$

Any nonequality $f(x) \neq g(x)$ can also be written as a collection of two strict inequalities:

$$f(x) > g(x); \quad f(x) < g(x).$$

Several systems of inequalities in one variable form a *collection of systems* of inequalities if a problem is to find all those values of the variable each of which satisfies at least one of the given systems.

Example 8. Solve the system of inequalities $\begin{cases} \dfrac{x^2+x-4}{x}<1 \\ x^2<64. \end{cases}$

Solution. Let us first consider the first inequality. We have:

$$\frac{x^2+x-4}{x}-1<0, \quad \frac{(x-2)(x+2)}{x}<0.$$

With the aid of the curve of signs (Fig. 14), we find the solution of this inequality: $(-\infty, -2) \cup (0, 2)$. Let us solve the second inequality of the given system. We have: $x^2-64 < 0$ or $(x-8)(x+8) < 0$.

With the aid of the curve of signs (Fig. 15), we find the solution of this inequality: $(-8, 8)$.

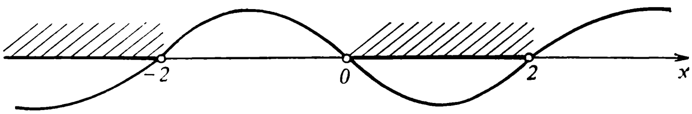

Fig. 14

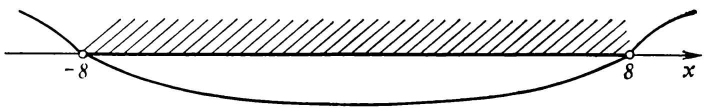

Fig. 15

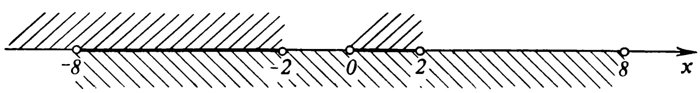

Fig. 16

Marking the found solutions of the first and second inequalities on the number line (Fig. 16), we find the intersection of the solutions. Answer: $(-8, -2) \cup (0, 2)$.

Example 9. Find the domain of definition of the function

$$f(x) = \sqrt{\frac{3x-6}{x+2}} + \sqrt[4]{(x^4-5x^3+6x^2)(1-x^2)}.$$

Solution. The problem is reduced to solving the following system of inequalities:

$$\begin{cases} \dfrac{3x-6}{x+2} \geqslant 0 \\ (x^4-5x^3+6x^2)(1-x^2) \geqslant 0. \end{cases}$$

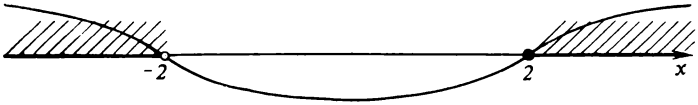

Fig. 17

We transform the first inequality of the system to $\frac{x-2}{x+2} \geqslant 0$ and, with the aid of the curve of signs shown in Fig. 17, find the solution of this inequality: $(-\infty, -2) \cup (2, \infty)$.

We then transform the second inequality of the system to:

$$x^2 (x - 2) (x - 3) (x - 1) (x + 1) \leqslant 0.$$

With the aid of the curve of signs shown in Fig. 18, we find the solution of this inequality: $[-1, 1] \cup [2, 3]$.

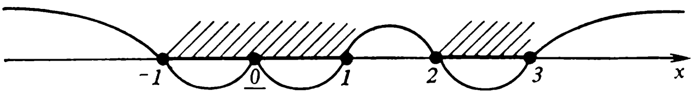

Fig. 18

Marking the found solutions of the first and second inequalities of the original system on the number line (Fig. 19), we find the intersection of the solutions: $[2, 3]$.

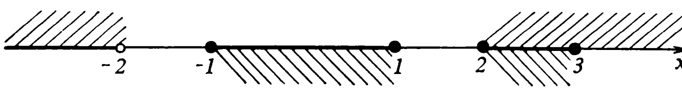

Fig. 19

Example 10. Solve the collection of inequalities

$$\left[\begin{array}{l} x^5 \geqslant 100x^3 \\ \dfrac{(x+9)(5x-x^2-18)}{x^2-18x+45} \geqslant 0. \end{array}\right.$$

Solution. Let us first transform the first inequality of the collection to: $x^3 (x - 10)(x + 10) \geqslant 0$. With the aid of the curve of signs

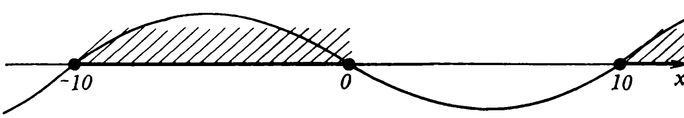

Fig. 20

shown in Fig. 20, we find the solution of this inequality: $[-10, 0] \cup [10, \infty)$.

Consider the second inequality of the collection. We have:

$$\frac{(x+9)(x^2-5x-18)}{(x-3)(x-15)} \leqslant 0.$$

Since the discriminant of the quadratic trinomial $x^2 - 5x - 18$ is negative, and the leading coefficient is positive, we have:

$$x^2 - 5x - 18 > 0$$

for all values of x, and, consequently, dividing both sides of the inequality by $x^2 - 5x - 18$ and retaining the sense of the inequality, we get the equivalent inequality:

$$\frac{x+9}{(x-3)(x-15)} \leqslant 0.$$

With the aid of the curve of signs represented in Fig. 21, we find the solution of the last inequality: $(-\infty, -9] \cup (3, 15)$.

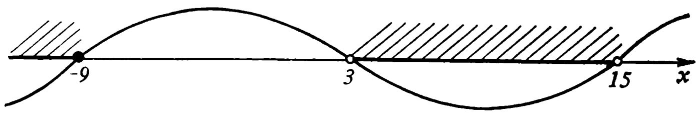

Fig. 21

Combining the found solutions of each of the inequalities entering the collection (Fig. 22), we get: $(-\infty, 0] \cup (3, +\infty)$, which is just the solution of the original collection.

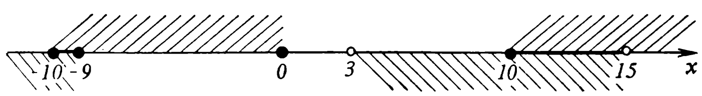

Fig. 22

Example 11. Solve the collection of systems of inequalities

$$\begin{cases} x-2>0 \\ x^2<16 \end{cases}; \quad \begin{cases} 3x-9<0 \\ 100 \geqslant x^2. \end{cases}$$

The solution of the first system is represented by the number interval $(2, 4)$; the solution of the second by the number interval $[-10, 3)$. With the aid of the number line (see Fig. 23), we get the union of

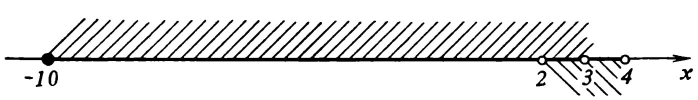

Fig. 23

the solutions of the first and second systems: $[-10, 4)$, that is, the solution of the given collection of systems.

Example 12. Find for what values of a both roots of the quadratic trinomial $(a-2)x^2 - 2ax + a + 3$ are positive.

Solution. Since, by hypothesis, the trinomial has real roots, its discriminant $D \geqslant 0$, that is, the inequality $4a^2 - 4(a-2)(a+3) \geqslant 0$ must be fulfilled.

By Viète's theorem, we have: $\begin{cases} x_1x_2 = \dfrac{a+3}{a-2} \\ x_1+x_2 = \dfrac{2a}{a-2}, \end{cases}$ where x_1 and x_2 are roots of the given quadratic trinomial. By hypothesis, both roots are positive, hence, $x_1x_2 > 0$, $x_1+x_2 > 0$.

Thus, we arrive at the following system of inequalities:

$$\begin{cases} 4a^2-4(a-2)(a+3) \geqslant 0 \\ \dfrac{a+3}{a-2} > 0 \\ \dfrac{2a}{a-2} > 0. \end{cases}$$

Solving this system, we get:

$\begin{cases} a \leqslant 6 \\ a \leqslant -3;\ a > 2 \\ a \leqslant 0;\ a > 2 \end{cases}$ whence we find: $a < -3$, $2 < a \leqslant 6$.

Example 13. Find out for what values of a the inequality

$$\frac{x^2-8x+20}{ax^2+2(a+1)x+9a+4} < 0$$

is fulfilled for any values of x.

Solution. The trinomial $x^2 - 8x + 20$ has a positive leading coefficient and a negative discriminant, hence, $x^2-8x+20 > 0$ for all x's, and therefore the denominator of the given fraction, that is, $ax^2 + 2(a+1)x + 9a + 4$, must be negative for all x's. This is possible if $a < 0$ and $D < 0$, where D is the discriminant of the trinomial $ax^2 + 2(a+1)x + 9a + 4$. Hence, the problem is reduced to solving the system of inequalities

$\begin{cases} a < 0 \\ 4(a+1)^2 - 4a(9a+4) < 0, \end{cases}$ from which we get $a < -\frac{1}{2}$.

4. Inequalities Containing the Variable Under the Modulus Sign. When solving inequalities containing the variable under the modulus sign, sometimes we can successfully apply Theorem 4 (see Item 1 of this section).

Let us, for instance, solve the inequality $|f(x)| > |g(x)|$. If $p(x)$ is a function, then $|p(x)| \geqslant 0$ and $|p(x)|^2 = (p(x))^2$.

This means that, by Theorem 4, the inequality $|f(x)| > |g(x)|$ is equivalent to the inequality $(f(x))^2 > (g(x))^2$. Besides, it is sometimes useful to take advantage of the geometric interpretation of the modulus of a real number. The thing is that $|a|$ denotes the distance from point a on the number line to the origin, while $|a-b|$ denotes the distance between points a and b.

Example 14. Solve the inequality $|x-1|<2$.

Solution. First Method. Since both sides of the given inequality are nonnegative for all x's, when squaring them, we get the inequality $(x-1)^2<4$ which is equivalent to the given inequality. We then have: $x^2-2x-3<0$, whence we find the solution: $(-1, 3)$.

Second Method. We may regard $|x-1|$ as the distance on the number line between points x and 1. Hence, we have to indicate on the number line all such points x which are at a distance less

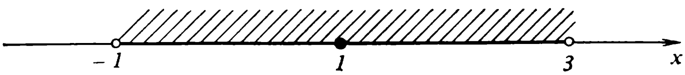

Fig. 24

than 2 (Fig. 24) from the point having the coordinate 1. The desired solution is: $(-1, 3)$.

Third Method. Since

$$|x-1|=\begin{cases} x-1 \text{ if } x-1\geqslant 0, \\ -(x-1) \text{ if } x-1<0, \end{cases}$$

the given inequality is equivalent to the collection of two systems:

$$\begin{cases} x-1\geqslant 0 \\ x-1<2 \end{cases}; \quad \begin{cases} x-1<0 \\ -(x-1)<2. \end{cases}$$

From the first system we get: $1\leqslant x<3$, from the second: $-1<x<1$. Combining these solutions, we find the solution of the given inequality, $(-1, 3)$.

Example 15. Solve the inequality $|2x-1|\leqslant|3x+1|$.

Solution. Squaring both sides of the inequality, we get:

$$(2x-1)^2\leqslant(3x+1)^2, \text{ and further } x(x+2)\geqslant 0,$$

whence we find: $(-\infty, -2]\cup[0, +\infty)$.

Example 16. Solve the inequality $\left|\frac{2x+3}{3x-2}\right|>1$.

Solution. This inequality is equivalent to

$$\left(\frac{2x+3}{3x-2}\right)^2>1,$$

which can be rewritten as follows:

$$\frac{4x^2+12x+9}{9x^2-12x+4}-1>0 \quad \text{or} \quad \frac{-5x^2+24x+5}{(3x-2)^2}>0,$$

whence $\dfrac{5\left(x+\frac{1}{5}\right)(x-5)}{9\left(x-\frac{2}{3}\right)^2}<0.$

Using the method of intervals (Fig. 25), we find the solution of the last and given inequality: $\left(-\frac{1}{5}, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 5\right)$.

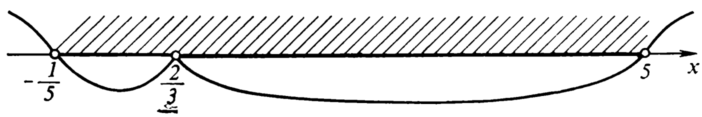

Fig. 25

Example 17. Solve the inequality $|x^2 - 3x + 2| \leqslant 2x - x^2$.

Solution. The given inequality is equivalent to the following collection of systems:

$$\begin{cases} x^2 - 3x + 2 \geqslant 0 \\ x^2 - 3x + 2 \leqslant 2x - x^2 \end{cases}; \qquad \begin{cases} x^2 - 3x + 2 < 0 \\ -(x^2 - 3x + 2) \leqslant 2x - x^2. \end{cases}$$

Solving this collection, we find:

$$\begin{cases} (x-1)(x-2) \geqslant 0 \\ \left(x - \frac{1}{2}\right)(x-2) \leqslant 0 \end{cases}; \qquad \begin{cases} (x-1)(x-2) < 0 \\ x - 2 \leqslant 0 \end{cases};$$

$$\begin{cases} x \leqslant 1;\ x \geqslant 2 \\ \frac{1}{2} \leqslant x \leqslant 2 \end{cases}; \qquad \begin{cases} 1 < x < 2 \\ x \leqslant 2, \end{cases}$$

whence $$\frac{1}{2} \leqslant x \leqslant 1;\ x = 2;\ 1 < x < 2.$$

Combining the found solutions, we get: $\left[\frac{1}{2}, 2\right]$.

Example 18. Solve the inequality $|x - 4| + |2x + 6| > 10$.

Solution. By the definition of modulus, we have: $|x - 4| = x - 4$ if $x \geqslant 4$, and $|x - 4| = -(x - 4)$ if $x < 4$. Hence, analysing the function $|x - 4|$, we have to consider two possibilities: $x \geqslant 4$, $x < 4$. Similarly, $|2x + 6| = 2x + 6$ if $x \geqslant -3$, and $|2x + 6| = -(2x + 6)$ if $x < -3$. Hence, analysing $|2x + 6|$ we also have two possibilities: $x \geqslant -3$; $x < -3$. Thus, we have to know the position of point x relative to points 4 and —3 on the coordinate line. These points divide the number line into the following three intervals: $(-\infty, -3]$, $[3, 4]$, $[4, \infty)$. Considering the given inequality on each of these intervals, we get the collection of three systems:

$$\begin{cases} x \leqslant -3 \\ -(x-4) - (2x+6) > 10 \end{cases}; \qquad \begin{cases} -3 \leqslant x \leqslant 4 \\ -(x-4) + (2x+6) > 10 \end{cases};$$

$$\begin{cases} x \geqslant 4 \\ (x-4) + (2x+6) > 10. \end{cases}$$

From the first system we find: $x < -4$, from the second: $0 < x \leqslant 4$, from the third: $x \geqslant 4$. Combining the found solutions, we get: $(-\infty, -4) \cup (0, +\infty)$.

5. Problems on Setting Up Inequalities.

Example 19. A boy had some number of stamps. He was presented with a stamp book. If he glues 20 stamps on each sheet, then some stamps will have no space to be glued on, and if he glues 23 stamps per sheet, then at least one sheet will be left empty. If the boy is presented with another stamp book with 21 stamps glued on each of its sheet, then he will have 500 stamps all in all. How many sheets has the stamp book?

Solution. Let us introduce two variables: x denoting the number of sheets in the book, y the number of stamps possessed by the boy.

If the boy glues 20 stamps on each sheet, then he will glue $20x$ stamps. This number is, by hypothesis, less than the number of stamps the boy had, that is, $20x < y$. If he glues 23 stamps per sheet, then it suffices to use $(x - 1)$ sheets for gluing the total number of stamps, that is, $23(x - 1)$ stamps. By hypothesis, this number is not less than the number of stamps possessed by the boy, that is, $23(x - 1) \geqslant y$. Finally, we know that if the boy is presented with a stamp book with $21x$ stamps glued on its sheets, then the total number of stamps will be 500, that is, $y + 21x = 500$. Thus, it is possible to write the following system:

$$\begin{cases} 20x < y \\ 23x - 23 \geqslant y \\ 21x + y = 500. \end{cases}$$

Expressing y from the equation of the above system and substituting the result into both inequalities of the system, we get the system of inequalities

$$\begin{cases} 20x < 500 - 21x \\ 23x - 23 \geqslant 500 - 21x. \end{cases}$$

Solving it we find: $\frac{523}{44} \leqslant x < \frac{500}{41}$.

By hypothesis, x is an integer. The indicated interval contains only one integer—12. Hence, the book has 12 sheets.

Example 20. It takes a raft 24 hours to cover the way from A to B, while a motor-boat covers the way from A to B and back for no less than 10 hours. If the speed of the boat (in stagnant water) is increased by 40%, then it will take it no more than 7 hours to cover the way from A to B and back. How much time does it take the boat to go from A to B, and how much time does it take it to return from B to A?

Solution. Let x denote in kilometres per hour the rate of flow of the river (and, consequently, the speed of motion of the raft), and let y be the speed in kilometres per hour of the boat in stagnant water. Then the way from A to B is equal to $24x$ km, and the time during which the motor-boat moves from A to B and returns back equals $\left(\frac{24x}{y+x}+\frac{24x}{y-x}\right)$ hours. If the speed of the boat becomes equal to $1.4y$ km/h, then it will take the boat $\left(\frac{24x}{1.4y+x}+\frac{24x}{1.4y-x}\right)$ hours to cover the distance from A to B and return back. According to the conditions of the problem, the first time is no less than 10 hours, and the second no more than 7 hours. Thus, we get the system of inequalities

$$\begin{cases} \dfrac{24x}{y+x}+\dfrac{24x}{y-x}\geqslant 10 \\ \dfrac{24x}{1.4y+x}+\dfrac{24x}{1.4y-x}\leqslant 7. \end{cases}$$

In each of the four fractions, we divide termwise both the numerator and denominator by x and introduce a new variable: $t=\frac{y}{x}$. Since, according to the sense of the problem, $y>x$, we have: $t>1$. Thus, we get the system of inequalities with respect to the variable t

$$\begin{cases} t>1 \\ \dfrac{24}{t+1}+\dfrac{24}{t-1}\geqslant 10 \\ \dfrac{24}{1.4t+1}+\dfrac{24}{1.4t-1}\leqslant 7. \end{cases}$$

Since $t>1$, the denominators of the fractions in the second and third inequalities of this system are positive. Therefore, getting rid of the denominators in these inequalities and carrying out all necessary transformations, we get:

$$\begin{cases} t>1 \\ 5t^2-24t-5\leqslant 0 \\ 49t^2-240t-25\geqslant 0, \end{cases} \quad \text{and further} \quad \begin{cases} t>1 \\ 5(t-5)\left(t+\frac{1}{5}\right)\leqslant 0 \\ 49(t-5)\left(t+\frac{5}{49}\right)\geqslant 0. \end{cases}$$

The solution of the last system is $t=5$. The problem requires to find the time of motion of the boat from A to B and from B to A. The time of motion from A to B is expressed by the fraction $\frac{24}{t+1}$, and, hence, is equal exactly to 4 hours. The time of motion from B to A is expressed by the fraction $\frac{24}{t-1}$, and is exactly 6 hours.

EXERCISES

In Problems 835 through 869, solve the given inequalities:

835. $x(x-1)^2>0$. 836. $(2-x)(3x+1)(2x-3)>0$.
837. $(3x-2)(x-3)^3(x+1)^3(x+2)^4<0$. 838. $x^2-25<0$.
839. $x^3-64x>0$. 840. $x^2+10\leqslant 7x$.
841. $x^2-7x<3$. 842. $-x^2-16+8x\geqslant 0$.
843. $x^2+5x+8>0$. 844. $x^4+8x^3+12x^2\geqslant 0$.
845. $(x-1)(x^2-3x+8)<0$. 846. $(x-1)(x^2-1)(x^3-1)(x^4-1)\leqslant 0$.
847. $\dfrac{(x-1)(3x-2)}{5-2x}>0$. 848. $\dfrac{(x+1)(x+2)(x+3)}{(2x-1)(x+4)(3-x)}>0$.
849. $(16-x^2)(x^2+4)(x^2+x+1)(x^2-x-3)\leqslant 0$.
850. $(x^2-4)(x^2-4x+4)(x^2-6x+8)(x^2+4x+4)<0$.
851. $(2x^2-x-5)(x^2-9)(x^2-3x)\leqslant 0$.
852. $\dfrac{x^2-5x+6}{x^2-12x+35}>0$. 853. $\dfrac{x^2-4x-2}{9-x^2}<0$.
854. $\dfrac{x^3+x^2+x}{9x^2-25}\geqslant 0$. 855. $\dfrac{x^4+x^2+1}{x^2-4x-5}<0$.
856. $\dfrac{x^3-x^2+x-1}{x+8}\leqslant 0$. 857. $\dfrac{x^4-2x^2-8}{x^2+x-1}<0$. 858. $\dfrac{3x-2}{2x-3}<3$.
859. $\dfrac{7x-4}{x+2}\geqslant 1$. 860. $\dfrac{1}{x}<\dfrac{1}{3}$. 861. $\dfrac{2x^2+18x-4}{x^2+9x+8}>2$.
862. $\dfrac{1}{x+1}+\dfrac{2}{x+3}>\dfrac{3}{x+2}$. 863. $\dfrac{x+1}{x-2}>\dfrac{3}{x-2}-\dfrac{1}{2}$.
864. $\dfrac{2}{x-1}-\dfrac{1}{x+1}>3$. 865. $\dfrac{1}{3x-2-x^2}>\dfrac{3}{7x-4-3x^2}$.
866. $\dfrac{3}{6x^2-x-12}<\dfrac{25x-47}{10x-15}-\dfrac{3}{3x+4}$. 867. $\dfrac{2-x}{x^3+x^2}\geqslant\dfrac{1-2x}{x^3-3x^2}$.
868. $\dfrac{1}{x+1}-\dfrac{2}{x^2-x+1}\leqslant\dfrac{1-2x}{x^3+1}$.
869. $\dfrac{10(5-x)}{3(x-4)}-\dfrac{11}{3}\times\dfrac{6-x}{x-4}\geqslant\dfrac{5(6-x)}{x-2}$.

In Problems 870 through 883, solve the indicated systems of inequalities:

870. $\begin{cases}\dfrac{3x+5}{7}+\dfrac{10-3x}{5}>\dfrac{2x+7}{3}-7\dfrac{1}{21}\\ \dfrac{7x}{3}-\dfrac{11(x+1)}{6}>\dfrac{3x-1}{3}-\dfrac{13-x}{2}.\end{cases}$

871. $\begin{cases}3-\dfrac{3-7x}{10}+\dfrac{x+1}{2}>4-\dfrac{7-3x}{2}\\ 7(3x-6)+4(17-x)>11-5(x-3).\end{cases}$

872. $\begin{cases}\dfrac{2x-11}{4}+\dfrac{19-2x}{2}<2x\\ \dfrac{2x+15}{9}>\dfrac{1}{5}(x-1)+\dfrac{x}{3}.\end{cases}$ 873. $\begin{cases}x^2-4x+3<0\\ 2x-4<0.\end{cases}$

874. $\begin{cases} 2x^2+2<5x \\ x^2 \geqslant x. \end{cases}$ 875. $\begin{cases} x^2<9 \\ x^2>7. \end{cases}$

876. $\begin{cases} \dfrac{x+3}{x-2}<1 \\ \dfrac{2x+3}{3x-2}<2. \end{cases}$ 877. $4x-2<x^2+1<4x+6.$

878. $\begin{cases} (2x+3)(2x+1)(x-1)<0 \\ (x+5)(x+1)(1-2x)(x-3)>0. \end{cases}$

879. $\begin{cases} (x^2+12x+35)(2x+1)(3-2x) \geqslant 0 \\ (x^2-2x-8)(2x-1) \geqslant 0. \end{cases}$

880. $\begin{cases} \dfrac{x+3}{3-x}<2 \\ x^3<16x \\ 4 \geqslant x^2. \end{cases}$ 881. $\begin{cases} \dfrac{(x+2)(x^2-3x+8)}{x^2-9} \leqslant 0 \\ \dfrac{1-x^2}{x^2+2x-8} \geqslant 0. \end{cases}$

882. $\dfrac{5x-7}{x-5}<4-\dfrac{x}{5-x}+\dfrac{3x}{x^2-25}<4.$

883. $\begin{cases} \dfrac{(x-1)^3(x^2-4)^2(x^2-9)^3(x^2+1)}{(1-3x)(x^2-x-6)(x^2-3x+16)}<0 \\ \dfrac{2x^2+x-16}{x^2+x}<1. \end{cases}$

In Problems 884 through 889, find the domain of definition for each of the given functions:

884. $f(x)=\sqrt[4]{\dfrac{x^2-6x-16}{x^2-12x+11}}+\dfrac{2}{\sqrt[3]{x^2-49}}.$

885. $f(x)=\sqrt{\dfrac{x^2-1}{3x-7-8x^2}}+\sqrt{4x^2-1}.$

886. $f(x)=\sqrt{\dfrac{(x-1)(x^2-x+1)}{x^3-1}}+\log(x^2-4x+4).$

887. $f(x)=\sqrt[6]{9-\left(\dfrac{4x-22}{x-5}\right)^2}+\dfrac{1}{\log_3(x-5)}.$

888. $f(x)=\sqrt[12]{\dfrac{x^3-2x^2+x-2}{x^2-4x+3}}+\sqrt{3x-5}.$

889. $f(x)=\log\dfrac{(x^2+4x+4)(4-x^2)}{x^2+2x+5}+\dfrac{1}{\sqrt{x-1}}+\sqrt[4]{8x^2-x^3-15x}.$

In Problems 890 through 894, solve the given collections of inequalities and systems of inequalities:

890. $(x-1)(x-2)(x-3)<0;\ x^2<1.$ 891. $\dfrac{3x-2}{x-3}>0;\ \dfrac{4x-1}{5x-2}<0.$

892. $x^2-5x+8 \leqslant 0;\ x^2-3x+6<0;\ x^2<1.$

893. $5x-20 \leqslant x^2 \leqslant 8x$; $1 < \frac{3x^2-7x+8}{x^2+1} < 2$.

894. $\begin{cases} x^2-5x+6>0 \\ \frac{3x-21}{x^2+x+4}<0 \end{cases}$; $\begin{cases} 2x+3>1 \\ \frac{1}{x}+\frac{1}{3}<0 \end{cases}$; $\begin{cases} \frac{x^2+9x-20}{11x-x^2-30} \leqslant -1 \\ x^2+18>5x. \end{cases}$

895. For what values of a does the quadratic trinomial $x^2+2(a+1)x+9a-5$ have: (a) no real roots; (b) only negative roots; (c) only positive roots?

896. For what values of a does the quadratic trinomial $(a^2-a-2)x^2+2ax+a^3-27$ have the roots of opposite signs?

897. For what values of a is the inequality $\frac{x^2+ax-1}{2x^2-2x+3}<1$ fulfilled for any x?

898. For what values of a is the system of inequalities $-6<\frac{2x^2+ax-4}{x^2-x+1}<4$ fulfilled for any x?

In Problems 899 through 938, solve the given inequalities:

899. $|x+5|>11$. 900. $|2x-5|<3$.
901. $|3x-1| \geqslant 5$. 902. $|2x-4| \leqslant 1$.
903. $|2x-1|<|4x+1|$. 904. $|1-3x|-|2x+3| \geqslant 0$.
905. $\left|-\frac{5}{x+2}\right|<\left|\frac{10}{x-1}\right|$. 906. $|1-2x|>3-x$.
907. $|x+8| \leqslant 3x-1$. 908. $|4-3x| \geqslant 2-x$.
909. $|2x-3| \geqslant 2x-3$. 910. $|5x^2-2x+1|<1$.
911. $|6x^2-2x+1| \leqslant 1$. 912. $|-2x^2+3x+5|>2$.
913. $\left|\frac{x+2}{2x-3}\right|<3$. 914. $\left|\frac{2x-3}{x^2-1}\right| \geqslant 2$. 915. $\left|\frac{x^2-3x+2}{x^2+3x+2}\right|>1$.
916. $\left|\frac{x^2-3x-1}{x^2+x+1}\right| \leqslant 3$. 917. $\left|\frac{x^2-5x+4}{x^2-4}\right| \geqslant 1$. 918. $x^2+2|x|-3 \leqslant 0$.
919. $x^2+5|x|-24>0$. 920. $|x^2-3x-15|<2x^2-x$.
921. $|x^2+x+10| \leqslant 3x^2+7x+2$.
922. $|2x^2+x+11|>x^2-5x+6$.
923. $|4x^2-9x+6|>-x^2+x-3$.
924. $\frac{|x-3|}{x^2-5x+6} \geqslant 2$. 925. $|x-6|>|x^2-5x+9|$.
926. $\frac{x^2-7|x|+10}{x^2-6x+9}<0$. 927. $\frac{x^2-|x|-12}{x-3} \geqslant 2x$.
928. $|x|+|x-1|<5$. 929. $|x+1|+|x-2|>5$.
930. $|2x+1|-|5x-2| \geqslant 1$.
931. $|3x-1|+|2x-3|-|x+5|<2$.
932. $|x-1|+|2-x|>3+x$.
933. $||2x+1|-5|>2$. 934. $||x-3|+1| \geqslant 2$.
935. $||x-1|+x|<3$. 936. $||x-2|-x+3|<5$.
937. $|2x-|3-x|-2| \leqslant 4$.
938. $\left|\frac{x^2-2x+1}{x^2-4x+4}\right|+\left|\frac{x-1}{x-2}\right|-12<0$.

939. More than 29 similar articles are contained in two boxes. The number of articles in the first box, less two articles, is more than three times the number of articles in the second box. Three times the number of articles in the first box is 60 articles or more than twice the number of articles in the second box. How many articles are there in each box?

940. There are more than 27 workers in two teams. The number of members in the first team is over twice the number of members in the second team, less twelve. The number of workers in the second team is nine times the number of workers of the first team, less ten. How many workers are there in each team?

941. If school children in a school are formed up into a column, eight abreast, then one row will turn out to be incomplete. If they are arranged seven abreast, then there will be two rows more, all of the rows being complete. If they are again rearranged, five abreast, then there will be another seven more rows, but one of the rows will be incomplete. How many school children are there in the school?

942. A certain amount of wire is wound on several 800-m reels, though one reel is not completely filled. The same happens if 900-m reels are used, though the total number of reels is three less. If the wire is wound on 1100-m reels, then the number of reels needed will be a further six less, but all the reels will be full. What is the length of the wire?

943. If a liquid is kept in large bottles (40-l capacity), then one bottle will not be filled. If the same quantity of liquid is stored in 50-l bottles, then five fewer bottles will be needed and all of them will be completely full. If the liquid is put into 70-l bottles, then a further four less bottles will be needed, but again one bottle will be not full. What is the volume of the liquid (in litres)?

944. Two teams with a total membership of 18 were to keep a twenty-four-hour watch for three days, one person at a time. For the first two days the first team was on duty, distributing their time equally among themselves. There were three girls in the second team, the rest being boys. For the time the second team was on duty the girls kept watch for one hour each, the rest of the time being equally distributed among the boys. It was found that the total number of hours each boy on the second team kept watch plus the time any member of the first team kept watch was less than nine hours. How many members were there in each team?

945. A sum of 10 roubles and 56 kopecks was paid for several textbooks and 56 kopecks for several exercise books. Six more textbooks than exercise books were bought. How many textbooks were purchased if the price of one textbook is over a rouble more than that of an exercise book?

946. A group of 30 students all took an examination. They were marked out of five, no one getting a 1. The sum of the marks they received was 93, the number of 3's being greater than the number of 5's and less than the number of 4's. In addition, the number of 4's was divisible by 10, and the number of 5's was even. How many of each mark were given to the students?

947. A group of students decided to buy a tape recorder for between 170 and 195 roubles. But at the last moment two students decided to opt out and therefore each of the remaining students had to pay one rouble more. How much did the tape recorder cost?

948. A first-grade article is more expensive than a second-grade article by as much as a second-grade article is more expensive than a third-grade article, but the difference in price is no more than 40% of the price of a second-grade article. A factory paid 9600 roubles for several first-grade articles and as much for several third-grade articles. The factory bought 1400 articles in all. The price of each grade of article was a round number of roubles. What is the price of a second-grade article?

SEC. 17. IRRATIONAL INEQUALITIES

We use the same techniques to solve irrational inequalities and irrational equations, that is, we raise both sides to the same natural power, we introduce new variables on both sides, and so on. But the difference of principle between the solution of irrational inequalities and that of irrational equations is that, when solving inequalities, a check by substitution is, as a rule, infeasible since the solution of an inequality is an infinite set. This means that when solving inequalities (and not only irrational ones), we must be sure that the transformations we do lead to equivalent inequalities.

Any irrational inequality containing a square-root of a variable is eventually reduced to an inequality of the form $\sqrt{f(x)} < g(x)$ or $\sqrt{f(x)} > g(x)$. Let us therefore discuss the question of solving the inequalities of the indicated form.

Consider the inequality

$$\sqrt{f(x)} < g(x). \tag{1}$$

It is clear that any solution of this inequality is at the same time a solution of the inequality $f(x) \geqslant 0$ (this condition defines the left-hand side of the inequality) and a solution of the inequality $g(x) > 0$ (since $g(x) > \sqrt{f(x)} \geqslant 0$).

Hence, Inequality (1) is equivalent to the system of inequalities:

$$\begin{cases} f(x) \geqslant 0 \\ g(x) > 0 \\ \sqrt{f(x)} < g(x), \end{cases}$$

where $f(x) \geqslant 0$ and $g(x) > 0$ are consequences of Inequality (1).

Since on the set defined by the first two inequalities of this system both sides of the third inequality take on only nonnegative values, squaring them on the indicated set is an equivalent transformation of Inequality (1). Thus, we conclude that Inequality (1) is equivalent to the system of inequalities

$$\begin{cases} f(x) \geqslant 0 \\ g(x) > 0 \\ f(x) < (g(x))^2. \end{cases}$$

Analogously, the inequality $\sqrt{f(x)} \leqslant g(x)$ is equivalent to the system of inequalities

$$\begin{cases} f(x) \geqslant 0 \\ g(x) \geqslant 0 \\ f(x) \leqslant (g(x))^2. \end{cases}$$

Let us now consider the inequality of the form

$$\sqrt{f(x)} > g(x). \tag{2}$$

It is equivalent to the system of inequalities $\begin{cases} f(x) \geqslant 0 \\ \sqrt{f(x)} > g(x) \end{cases}$, but in contrast to the preceding case, $g(x)$ may take on both nonnegative and negative values. Therefore, after considering System (2) in each of the two cases $g(x) < 0$ and $g(x) \geqslant 0$, we get the following collection of systems:

$$\begin{cases} g(x) < 0 \\ f(x) \geqslant 0 \\ \sqrt{f(x)} > g(x) \end{cases}; \quad \begin{cases} g(x) \geqslant 0 \\ f(x) \geqslant 0 \\ \sqrt{f(x)} > g(x). \end{cases}$$

The last inequality in the first of these systems may be omitted as a consequence of the first two inequalities, in the second system both sides of the last inequality may be squared.

Thus, Inequality (2) is equivalent to the collection of two systems of inequalities:

$$\begin{cases} g(x) < 0 \\ f(x) \geqslant 0 \end{cases}; \quad \begin{cases} g(x) \geqslant 0 \\ f(x) \geqslant 0 \\ f(x) > (g(x))^2. \end{cases}$$

Note that the second inequality of the second system may be omitted since it is a consequence of the last inequality of the system.

Example 1. Solve the inequality $\sqrt{2x-1} < x + 2$.

Solution. The given inequality is an inequality of the form (1). Therefore it is equivalent to the system of inequalities:

$$\begin{cases} 2x - 1 \geqslant 0 \\ x + 2 > 0 \\ 2x - 1 < (x + 2)^2, \end{cases}$$

that is, to the system

$$\begin{cases} x \geqslant \frac{1}{2} \\ x > -2 \\ x^2 + 2x + 5 > 0. \end{cases}$$

Since the quadratic trinomial $x^2 + 2x + 5$ has a negative discriminant and a positive leading coefficient, it is positive for all values of x. Therefore, the solution of the last system and, hence, of the given inequality is $\left[\frac{1}{2}, +\infty\right)$.

Example 2. Solve the inequality $\sqrt{(x+2)(x-1)} \geqslant 2(x+2)$.

Solution. Since the given inequality is an inequality of the form (2), it is equivalent to the collection of systems:

$$\begin{cases}(x+2)(x-1)\geqslant 0\\ 2(x+2)<0\end{cases};\quad \begin{cases}(x+2)(x-1)\geqslant 0\\ 2(x+2)\geqslant 0\\ (x+2)(x-1)\geqslant [2(x+2)]^2.\end{cases}$$

From the first system we find: $x<-2$, from the second: $x=-2$. Combining the solutions of the system of the collection, we get: $(-\infty, -2]$.

Example 3. Solve the inequality

$$\sqrt{3x}-\sqrt{2x+1}\geqslant 1. \tag{3}$$

Solution. Inequality (3) is equivalent to the following system:

$$\begin{cases}3x\geqslant 0\\ 2x+1\geqslant 0\\ \sqrt{3x}-\sqrt{2x+1}\geqslant 1.\end{cases} \tag{4}$$

It is advisable to rewrite the last inequality of System (4) in the form $\sqrt{3x}\geqslant 1+\sqrt{2x+1}$, where both sides are nonnegative, and therefore the squaring of both sides of this inequality is an equivalent transformation. Thus, from System (4) we pass to the following system which is equivalent to (4):

$$\begin{cases}x\geqslant 0\\ (\sqrt{3x})^2\geqslant(1+\sqrt{2x+1})^2\end{cases} \text{ or } \begin{cases}x\geqslant 0\\ \sqrt{2x+1}\leqslant\frac{x}{2}-1.\end{cases}$$

Further, we have: $\begin{cases}x\geqslant 0\\ \frac{x}{2}-1\geqslant 0\\ 2x+1\leqslant\left(\frac{x}{2}-1\right)^2,\end{cases}$

whence we get $[12, +\infty)$ which is the solution of the last system and, at the same time, of Inequality (3).

Example 4. Solve the inequality

$$\sqrt{2x+5}+\sqrt{x-1}>8. \tag{5}$$

Solution. Inequality (5) is equivalent to the system:

$$\begin{cases}2x+5\geqslant 0\\ x-1\geqslant 0\\ \sqrt{2x+5}+\sqrt{x-1}>8.\end{cases} \tag{6}$$

Since both sides of the last inequality of System (6) take on only nonnegative values, System (6) is equivalent to the following system:

$$\begin{cases} 2x+5\geqslant 0 \\ x-1\geqslant 0 \\ (\sqrt{2x+5}+\sqrt{x-1})^2>64 \end{cases} \quad \text{or} \quad \begin{cases} x\geqslant 1 \\ 2\sqrt{2x^2+3x-5}>60-3x. \end{cases} \tag{7}$$

The second inequality of System (7) is an inequality of the form (2), therefore System (7) is equivalent to the following collection of systems:

$$\begin{cases} x\geqslant 1 \\ 60-3x\geqslant 0 \\ x^2-372x+3620<0 \end{cases}; \qquad \begin{cases} x\geqslant 1 \\ 60-3x<0. \end{cases}$$

Note that for $x\geqslant 1$ the inequality $2x^2+3x-5\geqslant 0$ is true (since $2x^2+3x-5=(2x+5)(x-1)$), therefore the last collection of systems of inequalities is equivalent to the collection

$$\begin{cases} x\geqslant 1 \\ x\leqslant 20 \\ (x-10)(x-362)<0 \end{cases}; \qquad \begin{cases} x\geqslant 1 \\ x>20. \end{cases}$$

Solving this collection, we get: $10<x\leqslant 20$; $x>20$. Combining these solutions, we get: $(10, +\infty)$ which is the solution of Inequality (5).

Example 5. Solve the inequality

$$x^2+5x+4<5\sqrt{x^2+5x+28}. \tag{8}$$

Solution. Setting $y=\sqrt{x^2+5x+28}$, we find that $x^2+5x+4=y^2-24$. Then Inequality (8) is transformed to $y^2-5y-24<0$, and further, $(y-8)(y+3)<0$, whence we get: $-3<y<8$. We have obtained the following system of inequalities:

$$-3<\sqrt{x^2+5x+28}<8.$$

Since $\sqrt{x^2+5x+28}\geqslant 0$ for all permissible values of x, then the more so $\sqrt{x^2+5x+28}>-3$ for all x's from the domain of definition of Inequality (8), and therefore it suffices to solve the inequality

$$\sqrt{x^2+5x+28}<8.$$

This inequality is equivalent to the system $0\leqslant x^2+5x+28<64$. Since the inequality $x^2+5x+28\geqslant 0$ is fulfilled for any x's (the quadratic trinomial $x^2+5x+28$ has a negative discriminant and a positive leading coefficient), the last system is equivalent

to the inequality

$$x^2 + 5x - 36 < 0 \quad \text{or} \quad (x + 9)(x - 4) < 0,$$

whence we find: $(-9, 4)$ which is the solution of Inequality (8).

Example 6. Solve the inequality

$$\frac{1}{4}x > (\sqrt{1+x}-1)(\sqrt{1-x}+1). \tag{9}$$

Solution. Consider the function $\varphi(x) = \sqrt{1+x}+1$. Since $\varphi(x) > 0$ for any permissible values of x, multiplying both sides of Inequality (9) by $\varphi(x)$ and leaving the sense of Inequality (9) unchanged, we get the equivalent inequality $\frac{1}{4}x(\sqrt{1+x}+1) > (\sqrt{1+x}-1)(\sqrt{1-x}+1)(\sqrt{1+x}+1)$.

We further have:

$$\frac{1}{4}x(\sqrt{1+x}+1) > ((\sqrt{1+x})^2-1)(\sqrt{1-x}+1),$$

$$\frac{1}{4}x(\sqrt{1+x}+1) > x(\sqrt{1-x}+1),$$

$$x(\sqrt{1+x}+1-4(\sqrt{1-x}+1)) > 0,$$

$$x(\sqrt{1+x}-4\sqrt{1-x}-3) > 0. \tag{10}$$

Inequality (10) is equivalent to the collection of systems of inequalities:

$$\begin{cases} x > 0 \\ \sqrt{1+x} > 4\sqrt{1-x}+3 \end{cases}; \qquad \begin{cases} x < 0 \\ \sqrt{1+x} < 4\sqrt{1-x}+3, \end{cases}$$

which is, in turn, equivalent to the collection:

$$\begin{cases} x > 0 \\ 1+x \geqslant 0 \\ 1-x \geqslant 0 \\ \sqrt{1+x} > 4\sqrt{1-x}+3 \end{cases}; \qquad \begin{cases} x < 0 \\ 1+x \geqslant 0 \\ 1-x \geqslant 0 \\ \sqrt{1+x} < 4\sqrt{1-x}+3 \end{cases}$$

or

$$\begin{cases} 0 < x \leqslant 1 \\ \sqrt{1+x} > 4\sqrt{1-x}+3 \end{cases}; \qquad \begin{cases} -1 \leqslant x < 0 \\ \sqrt{1+x} < 4\sqrt{1-x}+3. \end{cases} \tag{11}$$

The first system of Collection (11) has no solution. Indeed if $0 < x \leqslant 1$, then $\sqrt{1+x} \leqslant \sqrt{2}$ and $\sqrt{1+x} < 3$. The more so $\sqrt{1+x} < 4\sqrt{1-x} + 3$, which contradicts the second inequality of the system. The second system of Collection (11) has the solution

$-1 \leqslant x < 0$, since it is easy to note that for these x's the second inequality of the system is true (indeed, if $-1 \leqslant x < 0$, then $\sqrt{1+x} < 1$, and then the more so $\sqrt{1+x} < 4\sqrt{1-x}+3$).

Thus, Inequality (9) has the following solution: $[-1, 0)$.

Example 7. Solve the inequality

$$\sqrt{x-2}+\sqrt{3-x} > \sqrt{x-1}-\sqrt{6-x}. \qquad (12)$$

Solution. This inequality is equivalent to the following system of inequalities:

$$\begin{cases} x-2 \geqslant 0 \\ 3-x \geqslant 0 \\ x-1 \geqslant 0 \\ 6-x \geqslant 0 \\ \sqrt{x-2}+\sqrt{3-x} > \sqrt{x-1}-\sqrt{6-x} \end{cases}$$

or

$$\begin{cases} 2 \leqslant x \leqslant 3 \\ \sqrt{x-2}+\sqrt{3-x} > \sqrt{x-1}-\sqrt{6-x}. \end{cases} \qquad (13)$$

Since $2 \leqslant x \leqslant 3$, we have: $x-1 \leqslant 2$, and therefore $\sqrt{x-1} \leqslant \sqrt{2}$. Further $6-x \geqslant 3$, therefore $\sqrt{6-x} \geqslant \sqrt{3}$.

Hence, $\sqrt{x-1}-\sqrt{6-x} \leqslant \sqrt{2}-\sqrt{3}$, and the more so $\sqrt{x-1}-\sqrt{6-x} < 0$.

But $\sqrt{x-2}+\sqrt{3-x} > 0$, consequently, the second inequality of System (13) is fulfilled for any permissible values of x from the domain of definition of Inequality (12), that is, System (13) and hence also Inequality (12) have the following solution: $[2, 3]$.

EXERCISES

In Problems 949 through 995, solve the given inequalities:

949. $\sqrt{2x+1} < 5$. 950. $\sqrt{3x-2} > 1$.

951. $\sqrt{\dfrac{x+3}{4-x}} \geqslant 2$. 952. $\sqrt{\dfrac{2x-1}{3x-2}} \leqslant 3$.

953. $\sqrt{2x+10} < 3x-5$. 954. $\sqrt{(x-3)(x+1)} > 3(x+1)$.

955. $\sqrt{(x+4)(2x-1)} < 2(x+4)$. 956. $\sqrt{(x+2)(x-5)} < 8-x$.

957. $\sqrt{x^2-x-12} < x$. 958. $\dfrac{\sqrt{17-15x-2x^2}}{x+3} > 0$.

959. $\sqrt{9x-20}<x$. **960.** $\sqrt{x^2-4x}>x-3$.

961. $\sqrt{3x^2-22x}>2x-7$. **962.** $\sqrt{x^2-5x+6}\leqslant x+4$.

963. $\sqrt{2x^2+7x+50}\geqslant x-3$. **964.** $\sqrt{x+1}-\sqrt{x-2}\leqslant 1$.

965. $\sqrt{x+3}-\sqrt{x-4}\geqslant 2$. **966.** $\sqrt{x-1}+\sqrt{x+2}\leqslant 1$.

967. $\sqrt{3x+1}+\sqrt{x-4}-\sqrt{4x+5}<0$. **968.** $2\sqrt{x+1}-\sqrt{x-1}\geqslant 2\sqrt{x-3}$.

969. $\sqrt{x-3}+\sqrt{1-x}>\sqrt{8x-5}$. **970.** $\sqrt{17-4x}+\sqrt{x-5}\leqslant\sqrt{13x+1}$.

971. $\sqrt{x+6}>\sqrt{x-1}+\sqrt{2x-5}$. **972.** $\sqrt{x-2}-\sqrt{x+3}-2\sqrt{x}\geqslant 0$.

973. $\sqrt{2\sqrt{7}+x}-\sqrt{2\sqrt{7}-x}>\sqrt[4]{28}$.

974. $x^2+\sqrt{x^2+11}<31$. **975.** $\dfrac{2}{x}-\dfrac{1}{2}>\sqrt{\dfrac{4}{x^2}-\dfrac{3}{4}}$.

976. $\dfrac{x-4}{\sqrt{x}+2}<x-8$. **977.** $\sqrt{\dfrac{2x-1}{x+2}}-\sqrt{\dfrac{x+2}{2x-1}}\geqslant\dfrac{7}{12}$.

978. $(x+5)(x-2)+3\sqrt{x(x+3)}>0$. **979.** $\sqrt{x^2-3x+5}+x^2\leqslant 3x+7$.

980. $2x^2-\sqrt{(x-3)(2x-7)}<13x+9$.

981. $\sqrt{2x+\sqrt{6x^2+1}}<x+1$. **982.** $(1+x^2)\sqrt{x^2+1}>x^2-1$.

983. $\sqrt[3]{x+5}+2>\sqrt[3]{x-3}$. **984.** $\sqrt[3]{1+\sqrt{x}}<2-\sqrt[3]{1-\sqrt{x}}$.

985. $\sqrt[4]{x-2}+\sqrt[4]{6-x}\geqslant\sqrt{2}$. **986.** $\sqrt{4-4x^3+x^6}>x-\sqrt[3]{2}$.

987. $\sqrt{x^4-2x^2+1}>1-x$. **988.** $\sqrt{3x^2+5x+7}-\sqrt{3x^2+5x+2}>1$.

989. $\dfrac{4}{\sqrt{2-x}}-\sqrt{2-x}<2$. **990.** $(x-3)\sqrt{x^2-4}\leqslant x^2-9$.

991. $\dfrac{6x}{x-2}-\sqrt{\dfrac{12x}{x-2}}-2\sqrt[4]{\dfrac{12x}{x-2}}>0$.

992. $\dfrac{2}{2+\sqrt{4-x^2}}+\dfrac{1}{2-\sqrt{4-x^2}}>\dfrac{1}{x}$.

993. $\dfrac{\sqrt{x^2-16}}{\sqrt{x-3}}+\sqrt{x-3}>\dfrac{5}{\sqrt{x-3}}$.

994. $\sqrt{x^2+3x+4}+\sqrt{x+1}>-3$. **995.** $\sqrt{x^2+3x+2}-\sqrt{x^2-x+1}<1$.

In Problems 996 through 998, solve the indicated equations:

996. $\sqrt{x+5-4\sqrt{x+1}}+\sqrt{x+2-2\sqrt{x+1}}=1$.

997. $\sqrt{x-2\sqrt{x-1}}+\sqrt{x+3-4\sqrt{x-1}}=1$.

998. $\sqrt{x+2+2\sqrt{x+1}}+\sqrt{x+2-2\sqrt{x+1}}=2$.

SEC. 18. EXPONENTIAL INEQUALITIES

An inequality of the form

$$a^{f(x)} > a^{g(x)},$$

where a is a positive number, different from 1, is called an *exponential inequality*. Its solution is based on the following theorems:

Theorem 1. *If $a > 1$, then the inequality $a^{f(x)} > a^{g(x)}$ is equivalent to the inequality $f(x) > g(x)$.*

Theorem 2. *If $0 < a < 1$, then the inequality $a^{f(x)} > a^{g(x)}$ is equivalent to the inequality $f(x) < g(x)$.*

Example 1. Solve the inequality

$$\sqrt[3]{2^{\frac{3x-1}{x-1}}} < 8^{\frac{x-3}{3x-7}}. \tag{1}$$

Solution. We transform Inequality (1) to

$$2^{\frac{3x-1}{3(x-1)}} < 2^{\frac{3(x-3)}{3x-7}}.$$

By Theorem 1, Inequality (1) is equivalent to the inequality

$$\frac{3x-1}{3(x-1)} < \frac{3(x-3)}{3x-7} \tag{2}$$

(Inequalities (1) and (2) are of the same sense.) From Inequality (2) we get:

$$\frac{3x-1}{3x-3} - \frac{3x-9}{3x-7} < 0, \qquad \frac{12x-20}{(3x-3)(3x-7)} < 0, \qquad \frac{x-\frac{5}{3}}{(x-1)\left(x-\frac{7}{3}\right)} < 0.$$

Solving the last inequality by the method of intervals, we get (Fig. 26): $(-\infty, 1) \cup \left(\frac{5}{3}, \frac{7}{3}\right)$ which is the solution of Inequality (1).

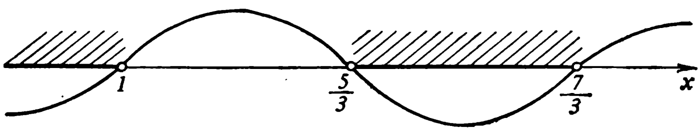

Fig. 26

Example 2. Solve the inequality

$$(0.04)^{5x-x^2-8} < 625. \tag{3}$$

Solution. Since $625 = (0.04)^{-2}$, the given inequality may be rewritten in the form

$$(0.04)^{5x-x^2-8} < (0.04)^{-2}.$$

By Theorem 2, Inequality (3) is equivalent to the inequality

$$5x - x^2 - 8 > -2 \tag{4}$$

(Inequalities (3) and (4) are of the opposite sense.) Solving Inequality (4), we get: (2, 3) which is the solution of Inequality (3).

Example 3. Solve the inequality

$$2^{x+2} - 2^{x+3} - 2^{x+4} > 5^{x+1} - 5^{x+2}. \tag{5}$$

Solution. We get:

$$2^{x+2}(1-2-2^2) > 5^{x+2}(5^{-1}-1),\quad 2^{x+2}(-5) > 5^{x+2}\left(-\frac{4}{5}\right),$$

$$\frac{2^{x+2}}{5^{x+2}} < \frac{4}{25} \text{ or } \left(\frac{2}{5}\right)^{x+2} < \left(\frac{2}{5}\right)^2.$$

The last inequality is equivalent to the inequality $x + 2 > 2$, whence we find: $(0, +\infty)$ which is the solution of Inequality (5).

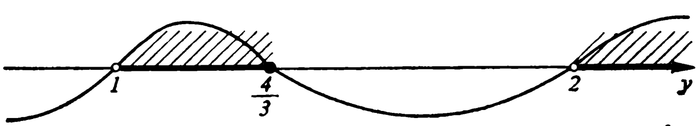

Fig. 27

Example 4. Solve the inequality

$$\frac{1}{(0.5)^x - 1} - \frac{1}{1-(0.5)^{x+1}} \geqslant 0.$$

Solution. Let us set $y = (0.5)^x$. Then the given inequality takes the form:

$$\frac{1}{y-1} - \frac{1}{1-0.5y} \geqslant 0,$$

whence, after transformations, we get:

$$\frac{y-\frac{4}{3}}{(y-1)(y-2)} \geqslant 0.$$

Using the method of intervals (Fig. 27), we find: $1 < y \leqslant \frac{4}{3}$; $y > 2$.

Thus, the problem has been reduced to solving the following collection:

$$1 < (0.5)^x \leqslant \frac{4}{3}; \quad (0.5)^x > 2$$

$$\text{or} \quad (0.5)^0 < (0.5)^x \leqslant (0.5)^{\log_{0.5}\frac{4}{3}}; \quad (0.5)^x > (0.5)^{-1}.$$

From the last collection we find: $(-\infty, -1) \cup \left[\log_{0.5}\frac{4}{3}, 0\right)$ which is the solution of the given inequality.

Example 5. Solve the inequality

$$8^x + 18^x - 2 \times 27^x > 0. \tag{6}$$

Solution. We rewrite Inequality (6) as follows:

$$(2^x)^3 + 2^x (3^x)^2 - 2 (3^x)^3 > 0,$$

and, setting $u = 2^x$, $v = 3^x$, we get a homogeneous inequality of the third degree:

$$u^3 + uv^2 - 2v^3 > 0. \tag{7}$$

Since $v = 3^x$, we have: $v > 0$, and therefore the division of both sides of Inequality (7) by v^3 (with the sign of Inequality (7) retained) is an equivalent transformation. As a result of the transformation, we get:

$$\left(\frac{u}{v}\right)^3 + \frac{u}{v} - 2 > 0.$$

Setting $z = \frac{u}{v}$, we get: $z^3 + z - 2 > 0$, and further $(z - 1)(z^2 + z + 2) > 0$, whence $z > 1$.

Thus, the problem has been reduced to solving the inequality

$$\frac{2^x}{3^x} > 1 \quad \text{or} \quad \left(\frac{2}{3}\right)^x > \left(\frac{2}{3}\right)^0.$$

From the last inequality we get: $(-\infty, 0)$ which is the solution of Inequality (6).

Example 6. Solve the inequality

$$(x^2 + x + 1)^x < 1. \tag{8}$$

Solution. Since the discriminant of the quadratic trinomial $x^2 + x + 1$ is negative and the coefficient of x^2 is positive, $x^2 + x + 1 > 0$ for all real values of x. Therefore the right-hand side of Inequality (8) can be represented as $(x^2 + x + 1)^0$, and Inequality (8) may be rewritten as follows:

$$(x^2 + x + 1)^x < (x^2 + x + 1)^0. \tag{9}$$

Neither Theorem 1 nor Theorem 2 can be applied to this inequality. Knowing that $x^2 + x + 1 > 0$, we do not know which is greater: $x^2 + x + 1$ or 1. For $x^2 + x + 1 > 1$ Theorem 1 is applicable to Inequality (9), while for $x^2 + x + 1 < 1$ Theorem 2 is applicable to it. Thus, two cases are posible: $0 < x^2 + x + 1 < 1$ or $x^2 + x + 1 > 1$.

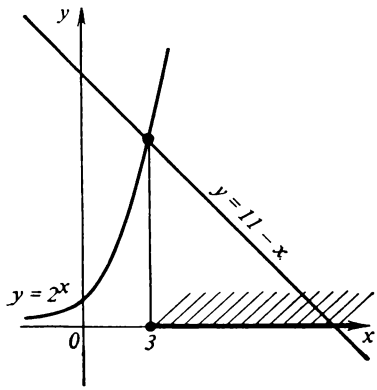

Fig. 28

Therefore Inequality (9) is equivalent to the following collection of systems of inequalities:

$$\begin{cases} x^2+x+1<1 \\ x>0 \end{cases}; \quad \begin{cases} x^2+x+1>1 \\ x<0, \end{cases}$$

or $$\begin{cases} x(x+1)<0 \\ x>0 \end{cases}; \quad \begin{cases} x(x+1)>0 \\ x<0. \end{cases}$$

The first system has no solution, and from the second system we get: $(-\infty, -1)$ which is the solution of Inequality (8).

Example 7. Solve the inequality

$$2^x \geqslant 11 - x.$$

Solution. The function $y = 2^x$ increases, and the function $y = 11 - x$ decreases throughout the number line.

It is clear that $x = 3$ is the root of the equation $2^x = 11 - x$. Then $[3, +\infty)$ is the solution of the given inequality (Fig. 28).

EXERCISES

In Problems 999 through 1031, solve the given inequalities:

999. $6^{3-x} < 216$. **1000.** $(\log 3)^{3x-7} > (\log_3 10)^{7x-3}$.

1001. $2^x \times 5^x > 0.1\,(10^{x-1})^5$. **1002.** $2^{x^2-6x-2.5} > 16\sqrt{2}$.

1003. $\left(\frac{1}{3}\right)^{-|x+2|} \geqslant 81$. **1004.** $(0.5)^{x-2} > 6$. **1005.** $(0.(4))^{x^2-1} > (0.(6))^{x^2+6}$.

1006. $0.3^{2+4+6+\ldots+2x} > 0.3^{72}$. **1007.** $\left(\frac{3}{5}\right)^{13x^2} \leqslant \left(\frac{3}{5}\right)^{x^4+36} < \left(\frac{25}{9}\right)^{-6x^2}$.

1008. $1 < 3^{|x^2-x|} < 9$. **1009.** $0.02^{1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+(-1)^n\frac{1}{2^n}+\ldots} < \sqrt[3]{0.02^{3x^2+5x}} < 1$.

1010. $\sqrt{3^{x-54}} - 7\sqrt{3^{x-58}} \leqslant 162$. **1011.** $8^{x+1} - 8^{2x-1} > 30$.

1012. $2^{2+x}-2^{2-x}>15.$ **1013.** $4^x-2^{2(x-1)}+8^{\frac{2}{3}(x-2)}>52.$

1014. $5^{2x+1}>5^x+4.$ **1015.** $\frac{1}{3^x+5}<\frac{1}{3^{x+1}-1}.$

1016. $5^{2\sqrt{x}}+5<5^{\sqrt{x}+1}+5^{\sqrt{x}}.$ **1017.** $36^x-2\times 18^x-8\times 9^x>0.$

1018. $4^{2x+1}+2^{2x+6}<4\times 8^{x+1}.$ **1019.** $4^{x+1.5}+9^x<9^{x+1}.$

1020. $2^{2x+2}+6^x-2\times 3^{2x+2}>0.$

1021. $\left(\frac{3}{2}\right)^{2x}+3\times\left(\frac{3}{2}\right)^{x-1}-\frac{1}{9}\times\left(\frac{2}{3}\right)^{x-2}+1.25>0.$

1022. $2^{4x}-2^{3x+1}-2^{2x}-2^{x+1}-2\leqslant 0.$

1023. $0.008^x+5^{1-3x}+0.04^{\frac{3}{2}(x+1)}<30.04.$

1024. $\sqrt{9^x-3^{x+2}}>3^x-9.$ **1025.** $25\times 2^x-10^x+5^x>25.$

1026. $|x-3|^{2x^2-7}>1.$ **1027.** $(4x^2+2x+1)^{x^2-x}>1.$

1028. $\sqrt{2(5^x+24)}-\sqrt{5^x-7}\geqslant\sqrt{5^x+7}.$

1029. $\sqrt{13^x-5}\leqslant\sqrt{2(13^x+12)}-\sqrt{13^x+5}.$

1030. $\frac{6-3^{x+1}}{x}>\frac{10}{2x-1}.$ **1031.** $\frac{2^{x+1}-7}{x-1}<\frac{10}{3-2x}.$

SEC. 19. LOGARITHMIC INEQUALITIES

Any inequality of the form

$$\log_a f(x)>\log_a g(x) \qquad (1)$$

is called a *logarithmic inequality*. Its solution is based on the following theorems.

Theorem 1. *If* $a>1$, *then Inequality* (1) *is equivalent to the system of inequalities*:

$$\begin{cases} f(x)>0\\ g(x)>0\\ f(x)>g(x). \end{cases} \qquad (2)$$

Theorem 2. *If* $0<a<1$, *then Inequality* (1) *is equivalent to the system of inequalities*:

$$\begin{cases} f(x)>0\\ g(x)>0\\ f(x)<g(x). \end{cases} \qquad (3)$$

Remarks. 1. When $a>1$, Inequality (1) and the last inequality of System (2) are of the same sense. When $0<a<1$, Inequality (1) and the last inequality of System (3) are of the opposite sense.

2. The first two inequalities of Systems (2) and (3) specify the domain of definition of Inequality (1).

3. The first inequality in System (2) may be omitted since it follows from the second and third. Analogously, the second inequality in System (3) may also be omitted.

Example 1. Solve the inequality

$$\log_{\frac{1}{2}} \frac{2x^2-4x-6}{4x-11} \leqslant -1. \tag{4}$$

Solution. Since $-1=\log_{\frac{1}{2}} 2$, Inequality (4) can be rewritten as follows:

$$\log_{\frac{1}{2}} \frac{2x^2-4x-6}{4x-11} \leqslant \log_{\frac{1}{2}} 2. \tag{5}$$

Here, the base of the logarithms $a=\frac{1}{2}$, i.e. $0<a<1$, and, consequently, by Theorem 2, Inequality (5) is equivalent to the following system:

$$\begin{cases} \dfrac{2x^2-4x-6}{4x-11}>0 \\ \dfrac{2x^2-4x-6}{4x-11}\geqslant 2. \end{cases}$$

The obtained system is equivalent to the inequality $\frac{2x^2-4x-6}{4x-11}\geqslant 2$, from which we find the solution of Inequality (4): $[2, 2.75) \cup [4, +\infty)$.

Example 2. Solve the inequality

$$\log_2 \frac{4}{x+3} > \log_2 (2-x).$$

Solution. By Theorem 1, the given inequality is equivalent to the following system of inequalities:

$$\begin{cases} \dfrac{4}{x+3}>0 \\ 2-x>0 \\ \dfrac{4}{x+3}>2-x, \end{cases}$$

whence we get:

$$\begin{cases} -3<x<2 \\ \dfrac{(x+2)(x-1)}{x+3}>0, \end{cases}$$

and further we have the solution of the given inequality: $(-3, -2) \cup (1, 2)$.

Example 3. Solve the inequality

$$\log_{0.2}(x^3+8) - 0.5\log_{0.2}(x^2+4x+4) \leqslant \log_{0.2}(x+58). \quad (6)$$

Solution. Inequality (6) is equivalent to the following system of inequalities:

$$\begin{cases} x^3+8>0 \\ x^2+4x+4>0 \\ x+58>0 \\ \log_{0.2}(x^3+8) - 0.5\log_{0.2}(x+2)^2 \leqslant \log_{0.2}(x+58). \end{cases}$$

Further, we have:

$$\begin{cases} x>-2 \\ x\neq -2 \\ x>-58 \\ \log_{0.2}(x^3+8) - \log_{0.2}\sqrt{(x+2)^2} \leqslant \log_{0.2}(x+58), \end{cases}$$

whence

$$\begin{cases} x>-2 \\ \log_{0.2}\dfrac{(x+2)(x^2-2x+4)}{|x+2|} \leqslant \log_{0.2}(x+58). \end{cases}$$

Since $x>-2$, we have: $|x+2| = x+2$, and

$$\begin{cases} x>-2 \\ \log_{0.2}(x^2-2x+4) \leqslant \log_{0.2}(x+58). \end{cases} \quad (7)$$

Finally, using Theorem 2 for the second inequality of System (7), we get the system:

$$\begin{cases} x>-2 \\ x^2-2x+4 \geqslant x+58, \end{cases} \quad \text{and further} \quad \begin{cases} x>-2 \\ x^2-3x-54 \geqslant 0, \end{cases}$$

whence we obtain: $[9, +\infty)$ which is the solution of Inequality (6).

Example 4. Solve the inequality

$$\log_{x-2}(2x-3) > \log_{x-2}(24-6x). \quad (8)$$

Solution. Neither Theorem 1 nor Theorem 2 may be applied to this logarithmic inequality since we do not know whether the base $(x-2)$ is greater or less than 1. If $x-2>1$, then Theorem 1 is applicable to Inequality (8); if $0<x-2<1$, then Inequality (8) is solved by using Theorem 2. Therefore, we have to consider two cases: (1) $x-2>1$; (2) $0<x-2<1$.

Thus, the problem has been reduced to solving the following collection of two systems of inequalities

$$\begin{cases} x-2>1 \\ 2x-3>0 \\ 24-6x>0 \\ 2x-3>24-6x \end{cases}; \qquad \begin{cases} 0<x-2<1 \\ 2x-3>0 \\ 24-6x>0 \\ 2x-3<24-6x. \end{cases}$$

From the first system we get: $\frac{27}{8}<x<4$, from the second: $2<x<3$. Thus, $(2,\ 3)\cup\left(\frac{27}{8},\ 4\right)$ is the solution of Inequality (8).

Example 5. Solve the inequality

$$\log_{x+\frac{5}{2}}\left(\frac{x-5}{2x-3}\right)^2<0. \tag{9}$$

Solution. We rewrite Inequality (9) as follows:

$$\log_{x+\frac{5}{2}}\left(\frac{x-5}{2x-3}\right)^2<\log_{x+\frac{5}{2}}1.$$

Reasoning as in the preceding example, we conclude that this inequality is equivalent to the following collection of systems of inequalities:

$$\begin{cases} x+\frac{5}{2}>1 \\ \left(\frac{x-5}{2x-3}\right)^2>0 \\ \left(\frac{x-5}{2x-3}\right)^2<1 \end{cases}; \qquad \begin{cases} 0<x+\frac{5}{2}<1 \\ \left(\frac{x-5}{2x-3}\right)^2>0 \\ \left(\frac{x-5}{2x-3}\right)^2>1 \end{cases}$$

or

$$\begin{cases} x>-1.5 \\ x\neq 5;\ x\neq 1.5 \\ \dfrac{(x+2)\left(x-\frac{8}{3}\right)}{(2x-3)^2}>0 \end{cases}; \qquad \begin{cases} -2.5<x<-1.5 \\ x\neq 5;\ x\neq 1.5 \\ \dfrac{(x+2)\left(x-\frac{8}{3}\right)}{(2x-3)^2}<0. \end{cases}$$

Solving this collection, we find the solution of Inequality (9): $(-2,\ -1.5)\cup\left(\frac{8}{3},\ 5\right)\cup(5,\ +\infty)$.

Example 6. Solve the inequality

$$\log_2^2(x-1)^2-\log_{0.5}(x-1)>5. \tag{10}$$

Solution. Since

$$\log_2 (x-1)^2 = 2 \log_2 | x-1 |$$

$$\text{and} \quad \log_{0.5} (x-1) = \frac{\log_2 (x-1)}{\log_2 0.5} = -\log_2 (x-1),$$

then Inequality (10) can be rewritten as:

$$4 \log_2^2 | x-1 | + \log_2 (x-1) > 5. \qquad (11)$$

Let us set $y = \log_2 (x-1)$. Since $x-1 > 0$ and hence $| x-1 | = x-1$, Inequality (11) takes the form: $4y^2 + y - 5 > 0$, whence we find: $y < -\frac{5}{4}$; $y > 1$.

Now, the problem is reduced to solving the collection of logarithmic inequalities:

$$\log_2 (x-1) < -\frac{5}{4}; \quad \log_2 (x-1) > 1$$

$$\text{or} \ \log_2 (x-1) < \log_2 2^{-\frac{5}{4}}; \quad \log_2 (x-1) > \log_2 2. \qquad (12)$$

From the first inequality of Collection (12) we get: $0 < x - 1 < 2^{-\frac{5}{4}}$, and, consequently, $1 < x < 1 + \frac{1}{2\sqrt[4]{2}}$.

From the second inequality of Collection (12) we get: $x-1 > 2$, that is, $x > 3$. Thus, $\left(1, 1 + \frac{1}{2\sqrt[4]{2}}\right) \cup (3, +\infty)$ is the solution of Inequality (10).

Example 7. Solve the inequality

$$x^{\log x} > 10. \qquad (13)$$

Solution. This inequality may be conditionally called exponential logarithmic. In Sec. 14, Item 2 when considering exponential logarithmic equations, we mentioned the possibility to use the method of taking the logarithms of both sides of the equations to the same base. The same is applied for solving exponential logarithmic inequalities. It is clear that the passage from the inequality $f(x) > g(x)$ to the inequality $\log_a f(x) > \log_a g(x)$ is possible only on the condition $f(x) > 0$, $g(x) > 0$, and $a > 1$, and the passage from the inequality $f(x) > g(x)$ to the inequality $\log_a f(x) < \log_a g(x)$ on the condition $f(x) > 0$, $g(x) > 0$, and $0 < a < 1$.

Let us return to Inequality (13). Both of its sides take on only positive values. Taking the logarithms of both sides of Inequality (13)

to the base 10, we get the inequality $\log x^{\log x} > \log 10$ which is equivalent to Inequality (13).

After transformations, we get: $\log x \cdot \log x > 1$, that is, $\log^2 x > 1$, whence $\log x < -1$, $\log x > 1$.

From the first inequality of the obtained collection we find: $0 < x < 0.1$, from the second: $x > 10$. Thus, $(0, 0.1) \cup (10, \infty)$ is the solution of Inequality (13).

Example 8. Solve the inequality

$$(8-x)^{\log_2^2 (8-x)} \leqslant 2^{3x-4}. \tag{14}$$

Solution. Taking the logarithms to the base 2 of both sides of Inequality (14), we get: $\log_2 (8-x)^{\log_2^2 (8-x)} \leqslant \log_2 2^{3x-4}$ which is equivalent to Inequality (14), and further, $\log_2^3 (8-x) \leqslant 3x-4$.

In the domain of definition of the inequality, that is, for $x < 8$, the function $y = \log_2^3 (8-x)$ decreases, while the function $y = 3x - 4$ increases. In addition, it is easy to notice that the equation $\log_2^3 (8-x) = 3x - 4$ has the root $x = 4$. Hence, Inequality (14) has the solution: $[4, 8)$ (Fig. 29).

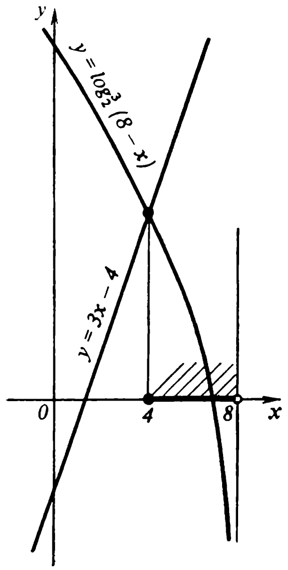

Fig. 29

EXERCISES

In Problems 1032 through 1116, solve the given inequalities:

1032. $\log_3 \frac{3}{x-1} > \log_3 (5-x)$. **1033.** $\log_{\frac{1}{4}} (2-x) > \log_{\frac{1}{4}} \frac{2}{x+1}$.

1034. $\log_{\frac{1}{2}} (5+4x-x^2) > -3$.

1035. $\log_{0.1} (x^2+75) - \log_{0.1} (x-4) \leqslant -2$.

1036. $\log_{\frac{1}{5}} (2x+5) < \log_{\frac{1}{5}} (16-x^2) - 1$.

1037. $\log_\pi (x+27) - \log_\pi (16-2x) < \log_\pi x$.

1038. $\frac{\log_{0.3} (x+1)}{\log_{0.3} 100 - \log_{0.3} 9} < 1$. **1039.** $2\log_8 (x-2) - \log_8 (x-3) > \frac{2}{3}$.

1040. $\frac{1}{2} + \log_9 x - \log_3 5x > \log_{\frac{1}{3}} (x+3)$. **1041.** $\log_{0.2}^2 (x-1) > 4$.

1042. $\log_2 ((x-3)(x+2)) + \log_{\frac{1}{2}} (x-3) < -\log_{\frac{1}{\sqrt{2}}} 3$.

1043. $\log_{\sqrt{2}} \frac{7-3x}{x+2} - \log_{\frac{1}{\sqrt{2}}} (x+2) > \log_{\frac{1}{2}} 4.$

1044. $\left(\frac{2}{5}\right)^{\log_{0.25}(x^2-5x+8)} \leqslant 2.5.$

1045. $2.25^{\log_2 (x^2-3x-10)} > \left(\frac{2}{3}\right)^{\log_{\frac{1}{2}}(x^2+4x+4)}.$

1046. $\left(\frac{1}{2}\right)^{\log_{\frac{1}{9}}(x^2-3x+1)} < 1.$ 1047. $\log_x (x-1) \geqslant 2.$

1048. $\log_x \sqrt{21-4x} > 1.$ 1049. $\log_x \frac{x+3}{x-1} > 1.$

1050. $\log_x (16-6x-x^2) \leqslant 1.$ 1051. $\log_{x^2-3} 729 > 3.$

1052. $\log_{\frac{x-1}{x+5}} 0.3 > 0.$ 1053. $\log_{|x-1|} 0.5 > 0.5.$

1054. $2^{\log_8 (x^2-6x+9)} \leqslant 3^{2\log_x \sqrt{x}-1}.$

1055. $\log_5 \sqrt{3x+4} \cdot \log_x 5 > 1.$ 1056. $\log_x (x^3+1) \cdot \log_{x+1} x > 2.$

1057. $\log_x (x+1) < \log_{\frac{1}{x}} (2-x).$ 1058. $\log_{|x-4|} (2x^2-9x+4) > 1.$

1059. $\log_{|x+6|} 2 \cdot \log_2 (x^2-x-2) \geqslant 1.$ 1060. $\log_{0.5}^2 x + \log_{0.5} x - 2 \leqslant 0.$

1061. $\frac{1-\log_4 x}{1+\log_2 x} \leqslant \frac{1}{2}.$ 1062. $\log_2 (x+1)^2 + \log_2 \sqrt{x^2+2x+1} > 6.$

1063. $(\log_2 x)^4 - \left(\log_{\frac{1}{2}} \frac{x^3}{8}\right)^2 + 9\log_2 \frac{32}{x^2} < 4\left(\log_{\frac{1}{2}} x\right)^2.$

1064. $\log_{\frac{1}{5}} x + \log_4 x > 1.$ 1065. $\log_x 5\sqrt{5} - 1.25 > (\log_x \sqrt{5})^2.$

1066. $\log_{\sqrt{2}} (5^x-1) \cdot \log_{\sqrt{2}} \frac{2\sqrt{2}}{5^x-1} > 2.$

1067. $2^{\log_{0.4} x \cdot \log_{0.4} 2.5x} > 1.$ 1068 $\sqrt{x^{\log_2 \sqrt{x}}} > 2.$

1069. $0.2^{6-\frac{3}{\log_4 x}} > \sqrt[3]{0.008^{2\log_4 x-1}}.$ 1070. $0.4^{\log_3 \frac{3}{x} \cdot \log_3 3x} > 6.25^{\log_3 x^2+2}.$

1071. $2^{\log_{0.5}^2 x} + x^{\log_{0.5} x} > 2.5.$ 1072. $3^{\log x+2} < 3^{\log x^2+5} - 2.$

1073. $9^{\log_2 (x-1)-1} - 8 \times 5^{\log_2 (x-1)-2} > 9^{\log_2 (x-1)} - 16 \times 5^{\log_2 (x-1)}.$

1074. $x^{\log_2 x} + 16x^{-\log_2 x} < 17.$ 1075. $\log_3 (4^x+1) + \log_{4^x+1} 3 > 2.5.$

1076. $\log_3 (3^x-1) \cdot \log_{\frac{1}{3}} (3^{x+2}-9) > -3.$

1077. $\log_2 (\log_3 (2-\log_4 x)) < 1.$ 1078. $x + \log(1+2^x) > x\log 5 + \log 6.$

1079. $\log_2 \left(9^x + 3^{2x-1} - 2^{x+\frac{1}{2}}\right) < x+3.5.$

1080. $\log_{\frac{1}{2}} x+\sqrt{1-4\log_{\frac{1}{2}}^2 x}<1$. 1081. $\sqrt{1-9\log_{\frac{1}{8}}^2 x}>1-4\log_{\frac{1}{8}} x$.

1082. $\log_2(x-1)-\log_2(x+1)+\log_{\frac{x+1}{x-1}} 2>0$.

1083. $\log_{\frac{x}{2}} 8+\log_{\frac{x}{4}} 8<\dfrac{\log_2 x^4}{\log_2 x^2-4}$. 1084. $\log_x 2\cdot\log_{2x} 2\cdot\log_2 4x>1$.

1085. $\log_2\log_{\frac{1}{2}}(x^2-2)<1$. 1086. $\left(\dfrac{1}{2}\right)^{\log_3\log_{\frac{1}{5}}\left(x^2-\frac{4}{5}\right)}\leqslant 1$.

1087. $0.3^{\log_{\frac{1}{3}}\log_2\frac{3x+6}{x^2+2}}>1$. 1088. $\log_3(\log_2(2-\log_4 x)-1)<1$.

1089. $\log_5\log_3\log_2(2^{2x}-3\times 2^x+10)>0$.

1090. $\log_2\left(1+\log_{\frac{1}{9}} x-\log_9 x\right)<1$. 1091. $\log_{\frac{1}{2}}\log_2\log_{x-1} 9>0$.

1092. $\log_3\log_{x^2}\log_{x^2} x^4>0$. 1093. $\log_x\log_2(4^x-12)\leqslant 1$.

1094. $\dfrac{\log_5(x^2+3)}{4x^2-16x}<0$. 1095. $\dfrac{3x^2-16x+21}{\log_{0.3}(x^2+4)}<0$.

1096. $\dfrac{(x-0.5)(3-x)}{\log_2|x-1|}>0$. 1097. $\dfrac{\log_{0.3}|x-2|}{x^2-4x}<0$.

1098. $\dfrac{\log 7-\log(-8-x^2)}{\log(x+3)}>0$. 1099. $\dfrac{\log_2(\sqrt{4x+5}-1)}{\log_2(\sqrt{4x+5}+11)}>\dfrac{1}{2}$.

1100. $\dfrac{\log_{0.5}(\sqrt{x+3}-1)}{\log_{0.5}(\sqrt{x+3}+5)}<\dfrac{1}{2}$. 1101. $\dfrac{\log\sqrt{x+7}-\log 2}{\log 8-\log(x-5)}<-1$.

1102. $\dfrac{\log(\sqrt{x+1}+1)}{\log\sqrt[3]{x-40}}<3$. 1103. $\log_5(x+3)\geqslant\log_{x+3} 625$.

1104. $\log_2 x\cdot\log_3 2x+\log_3 x\cdot\log_2 3x\geqslant 0$.

1105. $\log_{0.5}(x+2)\cdot\log_2(x+1)+\log_{x+1}(x+2)>0$.

1106. $\log_{\frac{1}{\sqrt{5}}}(6^{x+1}-36^x)\geqslant -2$. 1107. $\log_{\frac{\sqrt{3}}{3}}(2^{x+2}-4^x)\geqslant -2$.

1108. $25^{\log_5^2 x}+x^{\log_5 x}\leqslant 30$. 1109. $(2^x+3\times 2^{-x})^{2\log_2 x-\log_2(x+6)}>1$.

1110. $\dfrac{1}{\log_{0.5}\sqrt{x+3}}\leqslant\dfrac{1}{\log_{0.5}(x+1)}$. 1111. $\dfrac{1}{\log_2 x}\leqslant\dfrac{1}{\log_2\sqrt{x+2}}$.

1112. $\dfrac{\sqrt{\log_{0.5}^2 x-81}+2}{\log_{0.5} x-1}<1$. 1113. $|x-1|^{\log_2(4-x)}>|x-1|^{\log_2(1+x)}$.

1114. $\dfrac{x-1}{\log_3(9-3^x)-3}\leqslant 1$. 1115. $\dfrac{2+\log_3 x}{x-1}<\dfrac{6}{2x-1}$.

1116. $\dfrac{6}{2x+1}>\dfrac{1+\log_2(2+x)}{x}$.

In Problems 1117 and 1118, solve the given systems of inequalities:

1117. $\begin{cases} \log_x (x+2) > 2 \\ (x^2-8x+13)^{4x-6} < 1. \end{cases}$

1118. $\begin{cases} (x-1)\log 2 + \log (2^{x+1}+1) < \log (7 \times 2^x + 12) \\ \log_x (x+2) > 2. \end{cases}$

SEC. 20. PARAMETRIC EQUATIONS AND INEQUALITIES

Let there be given an equation

$$F(x, a) = 0. \tag{1}$$

If a problem is posed to find all the pairs (x, a) which satisfy the given equation, then we have an equation in two variables x and a. Another problem is also possible. If we fix the variable a, then Equation (1) may be considered as an equation in one variable x, the solutions of this equation being naturally determined by the chosen value of a. If for each value of a from a certain set of numbers A, we have to solve Equation (1) with respect to x, then Equation (1) is called an equation in one variable x and one *parameter* a, where the set A is the domain of change of the parameter. Let us agree that Equation (1) everywhere in this section is not an equation in two variables, but in one variable x and one parameter a.

Equation (1) describes briefly a family of equations resulting from Equation (1) for various concrete numerical values of the parameter a. Let, for instance, there be given the equation

$$2a(a-2)x = a-2 \tag{2}$$

and let the domain of change of the parameter $A = \{-1, 0, 1, 2, 3\}$. Then Equation (2) is a brief notation of the following family of equations:

$$\left\{\begin{array}{ll} 6x = -3 & \text{for } a = -1 \\ 0 \times x = -2 & \text{for } a = 0 \\ -2x = -1 & \text{for } a = 1 \\ 0 \times x = 0 & \text{for } a = 2 \\ 6x = 1 & \text{for } a = 3. \end{array}\right\}$$

Let us agree that the domain of change of the parameter here and elsewhere is the set of all real numbers (if no specific stipulations are made). And let us formulate the problem of solving an equation with a parameter in the following way: to solve Equation (1) with a parameter means to solve (on the set of real numbers) a family of equations resulting from Equation (1) for various real values of the parameter.

Since it is impossible to write out each equation of an infinite family, we usually try to find *singular* values of the parameter at which or when passing through which the equation is changed qualitatively. To clarify how to find singular values of the parameter, let us consider several examples.

Example 1. Solve the equation

$$2a(a-2)x = a-2. \tag{3}$$

Solution. Here, singular values of the parameter are those for which the coefficient of x vanishes, that is, $a = 0$ and $a = 2$. For these values of the parameter the division of both sides of the equation by the coefficient of x is impossible. If otherwise $a \neq 0$, $a \neq 2$, the division is possible. Hence it is appropriate to consider Equation (3) for the following values of the parameter:

(1) $a = 0$; (2) $a = 2$; (3) $\begin{cases} a \neq 0 \\ a \neq 2. \end{cases}$

(1) For $a = 0$ Equation (3) takes the form $0 \times x = -2$. This equation has no root.

(2) For $a = 2$ Equation (3) takes the form $0 \times x = 0$. Any real number serves as a root of this equation.

(3) If $a \neq 0$ and $a \neq 2$, then from Equation (3) we get: $x = \frac{a-2}{2a(a-2)}$, whence we find: $x = \frac{1}{2a}$.

Answer: (1) if $a = 0$, then there is no root; (2) if $a = 2$, then any real number is a root of Equation (3); (3) if $\begin{cases} a \neq 0 \\ a \neq 2, \end{cases}$ then $x = \frac{1}{2a}$.

Example 2. Solve the equation

$$(a-1)x^2 + 2(2a+1)x + (4a+3) = 0. \tag{4}$$

Solution. In this case $a = 1$ is a singular value. The thing is that for $a = 1$ Equation (4) is linear, and for $a \neq 1$ it is quadratic (this is just the above qualitative change). Hence, in solving Equation (4) it is expedient to consider the following cases: (1) $a = 1$; (2) $a \neq 1$.

(1) For $a = 1$ Equation (4) takes the form: $6x + 7 = 0$, whence we find: $x = -\frac{7}{6}$.

(2) For $a \neq 1$ we single out those values of the parameter for which the discriminant of Equation (4) vanishes.

The point is that the discriminant D can vanish for a certain value of the parameter $a = a_0$ and can change sign when passing through this point (for instance, $D < 0$ for $a < a_0$ and $D > 0$ for

$a > a_0$). Then the number of real roots of the quadratic equation changes (in our case, for $a < a_0$ there is no root, while for $a > a_0$ the equation has two roots). Hence, we may speak of a certain qualitative change in the equation. Therefore the values of the parameter for which $D = 0$ are usually referred to singular values as well. Let us form the discriminant D of Equation (4):

$$D = 4(2a+1)^2 - 4(a-1)(4a+3),$$

whence $D = 4(5a+4)$.

Equating the discriminant to zero, we find: $a = -\frac{4}{5}$ which is the second singular value of the parameter a. If $a < -\frac{4}{5}$, then $D < 0$; if $\begin{cases} a \geqslant -\frac{4}{5} \\ a \neq 1, \end{cases}$ then $D \geqslant 0$.

Hence, it remains to solve Equation (4) for each of the following two cases: $a < -\frac{4}{5}$; $\begin{cases} a \geqslant -\frac{4}{5} \\ a \neq 1. \end{cases}$

If $a < -\frac{4}{5}$, then Equation (4) has no real solution; if $\begin{cases} a \geqslant -\frac{4}{5} \\ a \neq 1, \end{cases}$ then we find: $x_{1,2} = \frac{-(2a+1) \pm \sqrt{5a+4}}{a-1}$.

Answer: (1) if $a < -\frac{4}{5}$, then there is no root; (2) if $a = 1$, then $x = -\frac{7}{6}$; (3) if $\begin{cases} a \geqslant -\frac{4}{5} \\ a \neq 1, \end{cases}$ then

$$x_{1,2} = \frac{-(2a+1) \pm \sqrt{5a+4}}{a-1}.$$

Example 3. Solve the equation

$$\frac{x^2+1}{a^2x-2a} - \frac{1}{2-ax} = \frac{x}{a}. \tag{5}$$

Solution. The first singular value of the parameter is the value $a = 0$. In this case Equation (5) has no root. Consider the case when $a \neq 0$. After being transformed, Equation (5) takes the form:

$$(1-a)x^2 + 2x + a + 1 = 0. \tag{6}$$

Equating the coefficient of x^2 to zero, we find the second singular value of the parameter: $a = 1$. For $a = 1$ Equation (6) takes the

form: $2x + 2 = 0$, whence we find: $x = -1$. If $a \neq 0$ and $a \neq 1$, then from the quadratic equation (6) we get: $x_1 = -1$, $x_2 = \frac{a+1}{a-1}$.

Check. When Equation (5) was replaced by Equation (6), there occurred an extension of the domain of definition of the equation, and, hence, extraneous roots might appear, namely, such values of x for which the denominator of a fraction in Equation (5) vanishes. In our example we have only one such value: $x = \frac{2}{a}$. It may happen so that for some value of the parameter a, x_1 will be equal to $\frac{2}{a}$. Then x_1 will be an extraneous root at this value a. It may also happen

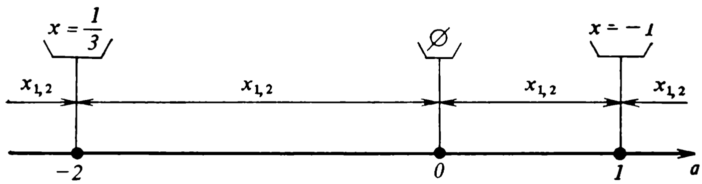

Fig. 30

that at some value a, x_2 will be equal to $\frac{2}{a}$. Then x_2 will be an extraneous root at this value a. Thus, let us find for what values of the parameter the equality $x_1 = \frac{2}{a}$ is fulfilled.

Setting $\frac{2}{a} = -1$, we find: $a = -2$. This means that if $a = -2$, then $x_1 = -1$ is an extraneous root. In this case $x_2 = \frac{a+1}{a-1} = \frac{-2+1}{-2-1} = \frac{1}{3}$.

Let us find the values of the parameter at which $x_2 = \frac{2}{a}$. Let $\frac{a+1}{a-1} = \frac{2}{a}$. Then $a^2 - a + 2 = 0$.

The last equation has no real root. This means that $x_2 = \frac{a+1}{a-1}$ is not an extraneous root for any value of the parameter.

We have made a check for the cases $a \neq 0$, $a \neq 1$. If $a = 0$, then, as was noted above, Equation (5) has no root. If $a = 1$, then Equation (5) has the root $x = -1$. Since for $a = 1$ and $x = -1$ the equality $x = \frac{2}{a}$ is not fulfilled, the root $x = -1$, found in the case of $a = 1$, is not extraneous.

Answer: (1) if $a=0$, then there is no root; (2) if $a=1$, then $x=-1$, (3) if $a=-2$, then $x=\frac{1}{3}$; (4) if $\begin{cases} a\neq 2 \\ a\neq 0 \\ a\neq 1, \end{cases}$ then $x_1=-1$, $x_2=\frac{a+1}{a-1}$.

The answer is illustrated graphically in Fig. 30.

Example 4. Solve the equation

$$\sqrt{x}+\sqrt{a}=\sqrt{1-(x+a)}. \tag{7}$$

Solution. Here $a=0$ is a singular value of the parameter (for $a<0$ the left-hand member of the equation is not defined, while for $a\geqslant 0$ it is defined). Therefore, it is appropriate to consider the following cases: (1) $a<0$; (2) $a\geqslant 0$.

(1) It is clear that for $a<0$ Equation (7) has no root.

(2) If $a\geqslant 0$, then, squaring both sides of Equation (7), we get the equation

$$2\sqrt{ax}=1-2x-2a. \tag{8}$$

No new singular values of the parameter are discovered here. Again, squaring both sides of the equation, we get the quadratic equation

$$4x^2+4(a-1)x+4a^2-4a+1=0. \tag{9}$$

Let us form the discriminant of Equation (9): $\frac{D}{4}=4(a-1)^2-4(4a^2-4a+1)$.

Equating it to zero, we find: $a_1=0, a_2=\frac{2}{3}$ which are other singular values of the parameter. Note that $D<0$ if $a>\frac{2}{3}$ (remember that we consider the case $a\geqslant 0$). Thus, it is appropriate to consider the following cases: $a>\frac{2}{3}$; $0\leqslant a\leqslant\frac{2}{3}$.

In the first case Equation (9) has no solution, in the second we get:

$$x_{1,2}=\frac{1-a\pm\sqrt{2a-3a^2}}{2}.$$

We noted above that for $a<0$ Equation (7) has no root. Thus, solving Equation (7), we get the following result: if $a<0$, $a>\frac{2}{3}$, then there is no root; if $0\leqslant a\leqslant\frac{2}{3}$, then the roots of Equation (7)

may be represented by the values

$$x_{1,2}=\frac{1-a\pm\sqrt{2a-3a^2}}{2}.$$

This limited formulation is connected with the fact that in the course of solving Equation (7) we squared both sides of this equation which might lead to the occurrence of extraneous roots. Hence, the found values x_1 and x_2 must be checked.

A check of these values by substituting them into Equation (7) involves much troubles, therefore we use another method. Note that the domain of definition of Equation (7) is given by the system of inequalities:

$$\begin{cases} x\geqslant 0 \\ 1-(x+a)\geqslant 0. \end{cases}$$

Further it follows from Equation (8) that the inequality $1-2x-2a\geqslant 0$ must be fulfilled. Hence, the roots of Equation (7) must satisfy the system of inequalities:

$$\begin{cases} x\geqslant 0 \\ 1-(x+a)\geqslant 0 \\ 1-2x-2a\geqslant 0 \end{cases} \text{ or } \begin{cases} x\geqslant 0 \\ x+a\leqslant 1 \\ x+a\leqslant \frac{1}{2}, \end{cases} \text{ whence } \begin{cases} x\geqslant 0 \\ x+a\leqslant \frac{1}{2}. \end{cases} \tag{10}$$

Let us check whether the value x_1 satisfies System (10). Consider the system of inequalities:

$$\begin{cases} \dfrac{1-a+\sqrt{2a-3a^2}}{2}\geqslant 0 \\ \dfrac{1-a+\sqrt{2a-3a^2}}{2}+a\leqslant \dfrac{1}{2}. \end{cases}$$

The second inequality of this system is equivalent to the inequality $\sqrt{2a-3a^2}\leqslant -a$, which in our case $\left(0\leqslant a\leqslant \frac{2}{3}\right)$ has the only solution: $a=0$.

Since this value also satisfies the first inequality of the system, the system under consideration has the only solution: $a=0$. This means that $x_1=\frac{1-a+\sqrt{2a-3a^2}}{2}$ is a root of Equation (7) for $a=0$ $\left(\text{for } a=0 \text{ we have: } x_1=\frac{1}{2}\right)$ and x_1 is an extraneous root if $a\neq 0$.

Let us check whether the value x_2 satisfies System (10). Consider the system of inequalities:

$$\begin{cases} \dfrac{1-a-\sqrt{2a-3a^2}}{2} \geqslant 0 \\ \dfrac{1-a-\sqrt{2a-3a^2}}{2}+a \leqslant \dfrac{1}{2}. \end{cases}$$

It is equivalent to the following system: $\begin{cases} \sqrt{2a-3a^2} \leqslant 1-a \\ \sqrt{2a-3a^2} \geqslant a, \end{cases}$

and further $\begin{cases} 4a^2-4a+1 \geqslant 0 \\ 4a^2-2a \leqslant 0 \end{cases}$ or $\begin{cases} (2a-1)^2 \geqslant 0 \\ 4a\left(a-\frac{1}{2}\right) \leqslant 0, \end{cases}$

whence $0 \leqslant a \leqslant \frac{1}{2}$. Thus, $x_2 = \frac{1-a-\sqrt{2a-3a^2}}{2}$ is a root of Equation (7) if the parameter a satisfies the following system:

$$\begin{cases} 0 \leqslant a \leqslant \dfrac{2}{3} \\ 0 \leqslant a \leqslant \dfrac{1}{2} \end{cases}, \text{ that is, } 0 \leqslant a \leqslant \frac{1}{2}.$$

Thus, the solution of Equation (7) can be written in the following way: (1) if $a<0$; $a>\frac{1}{2}$, then there is no root; (2) it $a=0$, then $x_1 = \frac{1-a+\sqrt{2a-3a^2}}{2}$; $x_2 = \frac{1-a-\sqrt{2a-3a^2}}{2}$; (3) if $0<a\leqslant\frac{1}{2}$, then $x = \frac{1-a-\sqrt{2a-3a^2}}{2}$.

Note that if $a = 0$, then $x_1 = x_2$. This enables us to write the answer more briefly.

Answer: (1) if $a<0$; $a>\frac{1}{2}$, then there is no root; (2) if $0\leqslant a\leqslant\frac{1}{2}$, then $x = \frac{1-a-\sqrt{2a-3a^2}}{2}$.

Example 5. Solve the system of equations

$$\begin{cases} x^3 = 2ax + ay \\ y^3 = ax + 2ay. \end{cases} \qquad (11)$$

Solution. Replacing the first equation of System (11) by the sum of its equations, and the second equation by their difference, we get

a system equivalent to the original

$$\begin{cases} x^3+y^3=3a\,(x+y) \\ x^3-y^3=a\,(x-y) \end{cases}.$$

or

$$\begin{cases} (x+y)\,(x^2-xy+y^2-3a)=0 \\ (x-y)\,(x^2+xy+y^2-a)=0. \end{cases}$$

The last system is equivalent to the following collection of four systems:

$$\left[\begin{array}{l} \begin{cases} x+y=0 \\ x-y=0 \end{cases} \qquad (12) \\ \begin{cases} x+y=0 \\ x^2+xy+y^2=a \end{cases} \qquad (13) \\ \begin{cases} x^2-xy+y^2=3a \\ x-y=0 \end{cases} \qquad (14) \\ \begin{cases} x^2-xy+y^2=3a \\ x^2+xy+y^2=a. \end{cases} \qquad (15) \end{array}\right.$$

From System (12) we find:

$$\begin{cases} x_1=0 \\ y_1=0. \end{cases}$$

This is the solution of System (11) for any values of $a \in \boldsymbol{R}$.
From System (13) we get:

$$\begin{cases} y=-x \\ x^2=a. \end{cases} \qquad (16)$$

Here $a=0$ is a singular value of the parameter. For $a<0$ the system has no real solution, and if $a \geqslant 0$, then we get:

$$\begin{cases} x_2=\sqrt{a} \\ y_2=-\sqrt{a} \end{cases}; \qquad \begin{cases} x_3=-\sqrt{a} \\ y_3=\sqrt{a}. \end{cases}$$

From System (14) we find:

$$\begin{cases} y=x \\ x^2=3a. \end{cases}$$

Here, as in the preceding case, $a=0$ is a singular value of the parameter. For $a<0$ the system has no real solution, and if $a \geqslant 0$, then we get:

$$\begin{cases} x_4=\sqrt{3a} \\ y_4=\sqrt{3a} \end{cases}; \qquad \begin{cases} x_5=-\sqrt{3a} \\ y_5=-\sqrt{3a}. \end{cases}$$

System (15) is symmetric. Setting $\begin{cases} x+y=u \\ xy=v, \end{cases}$ we get:

$$\begin{cases} u^2-3v=3a \\ u^2-v=a, \end{cases} \quad \text{whence} \quad \begin{cases} u=0 \\ v=-a. \end{cases}$$

Thus, we have obtained the following system of equations:

$$\begin{cases} x+y=0 \\ xy=-a \end{cases} \quad \text{or} \quad \begin{cases} y=-x \\ x^2=a. \end{cases}$$

This system coincides with System (16) which has been solved.

Answer: (1) if $a \leqslant 0$, then $(0, 0)$; (2) if $a > 0$, then $(0, 0)$; $(\sqrt{a}, -\sqrt{a})$, $(-\sqrt{a}, \sqrt{a})$; $(\sqrt{3a}, \sqrt{3a})$, $(-\sqrt{3a}, -\sqrt{3a})$.

Example 6. Solve the inequality

$$\frac{7x-11}{a+3} > (1+3a)\frac{x}{4}. \tag{17}$$

Solution. Setting $a+3=0$, we find: $a=-3$ which is the first singular value of the parameter. Hence, we have to consider the following cases: (1) $a < -3$; (2) $a=-3$; (3) $a > -3$.

(1) Consider the case $a < -3$. In this case $a+3 < 0$, and Inequality (17) is equivalent to the inequality $4(7x-11) < (a+3)\times(1+3a)x$, that is, to the inequality:

$$(3a^2+10a-25)x > -44. \tag{18}$$

Setting $3a^2+10a-25=0$, we find other singular values of the parameter: $a=\frac{5}{3}$; $a=-5$.

Thus, the solution of Inequality (18) has to be considered in the following cases:

$$\begin{cases} a<-5;\ a>\frac{5}{3} \\ a<-3 \end{cases}; \quad \begin{cases} a=-5;\ a=\frac{5}{3} \\ a<-3 \end{cases}; \quad \begin{cases} -5<a<\frac{5}{3} \\ a<-3, \end{cases}$$

that is, in the cases: $a < -5$; $a=-5$; $-5 < a < -3$.

In the first case $3a^2+10a-25 > 0$, and from Inequality (18) we find: $x > -\frac{44}{3a^2+10a-25}$.

In the second case Inequality (18) takes the form: $0\times x > -44$ which is true for any x. Finally, if $-5 < a < -3$, then $3a^2+10a-25 < 0$, and from Inequality (18) we find that $x < -\frac{44}{3a^2+10a-25}$.

(2) Consider the case $a=-3$. In this case Inequality (17) has no solution.

(3) Consider the case $a > -3$. In this case $a + 3 > 0$, and Inequality (17) is equivalent to the inequality

$$4(7x - 11) > (a + 3)(1 + 3a)x \text{ or}$$
$$(3a^2 + 10a - 25)x < -44. \tag{19}$$

The same as for Inequality (18), here the singular values of the parameter a are $\frac{5}{3}$ and -5. Since we now consider the case $a > -3$, we have to take into account only one of the indicated two singular values of the parameter: $a = \frac{5}{3}$. Thus, when solving Inequality (19), we must consider the following cases: $a > \frac{5}{3}$; $a = \frac{5}{3}$; $-3 < a < \frac{5}{3}$.

In the first case we find: $x < -\frac{44}{3a^2+10a-25}$, in the second Inequality (19) has no solution, and in the third we get: $x > -\frac{44}{3a^2+10a-25}$.

Answer: (1) if $a = -3$; $a = \frac{5}{3}$, then the inequality has no solution; (2) if $a < -5$; $-3 < a < \frac{5}{3}$, then $x > -\frac{44}{3a^2+10a-25}$; (3) if $-5 < a < -3$; $a > \frac{5}{3}$, then $x < -\frac{44}{3a^2+10a-25}$; (4) if $a = -5$, then $-\infty < x < +\infty$.

Example 7. Solve the inequality

$$ax^2 - 2x + 4 > 0. \tag{20}$$

Solution. Equating to zero the coefficient of x^2 and the discriminant of the quadratic trinomial $ax^2 - 2x + 4$, we find the first singular value of the parameter $a = 0$ and the second singular value $a = \frac{1}{4}$ $\left(\text{and if } a > \frac{1}{4}, \text{ then } D < 0; \text{ and if } a \leqslant \frac{1}{4}, \text{ then } D \geqslant 0\right)$. Let us solve Inequality (20) in each of the following four cases:

$$(1)\ a > \tfrac{1}{4};\quad (2)\ 0 < a \leqslant \tfrac{1}{4};\quad (3)\ a = 0;\quad (4)\ a < 0.$$

(1) If $a > \frac{1}{4}$, then the trinomial $ax^2 - 2x + 4$ has a negative discriminant and a positive leading coefficient. Hence, the trinomial is positive for any x, that is, the solution of Inequality (20) in this case is represented by the set of all real numbers.

(2) If $0 < a \leqslant \frac{1}{4}$, then the trinomial $ax^2 - 2x + 4$ has the following roots:

$$x_{1,2} = \frac{1 \pm \sqrt{1-4a}}{a}, \text{ where } \frac{1-\sqrt{1-4a}}{a} \leqslant \frac{1+\sqrt{1-4a}}{a}.$$

Hence, the solution of Inequality (20) is represented by the following collection:

$$x < \frac{1-\sqrt{1-4a}}{a}; \quad x > \frac{1+\sqrt{1-4a}}{a}.$$

(3) If $a=0$, then Inequality (20) takes the form: $-2x+4>0$, whence we get: $x<2$.

(4) If $a<0$, then we have: $\frac{1+\sqrt{1-4a}}{a} < \frac{1-\sqrt{1-4a}}{a}$.

Hence, in this case the solution of Inequality (20) is represented by the following system:

$$\frac{1+\sqrt{1-4a}}{a} < x < \frac{1-\sqrt{1-4a}}{a}.$$

Answer: (1) if $a>\frac{1}{4}$, then $-\infty<x<+\infty$; (2) if $0<a\leqslant\frac{1}{4}$, then $x<\frac{1-\sqrt{1-4a}}{a}$; $x>\frac{1+\sqrt{1-4a}}{a}$; (3) if $a=0$, then $x<2$; (4) if $a<0$, then $\frac{1+\sqrt{1-4a}}{a}<x<\frac{1-\sqrt{1-4a}}{a}$.

Example 8. Solve the inequality

$$\frac{x^2+1}{a^2x-2a} - \frac{1}{2-ax} > \frac{x}{a}. \tag{21}$$

Solution. Transform Inequality (21) to the form $\frac{x^2+1}{a(ax-2)}+\frac{1}{ax-2}-\frac{x}{a}>0$, and further

$$\frac{(1-a)x^2+2x+1+a}{a^2\left(x-\frac{2}{a}\right)}>0 \tag{22}$$

Inequality (22) is equivalent to Inequality (21). The value $a=0$ is the first singular value of the parameter. Equating the coefficient of x^2 in the numerator to zero, we find the second singular value of the parameter: $a=1$. Finally, the discriminant of the quadratic trinomial $(1-a)x^2+2x+1+a$ is equal to a^2. It vanishes for the already indicated singular value $a=0$.

Hence, it is appropriate to consider the following cases: (1) $a=1$; (2) $a=0$; (3) $\begin{cases} a\neq 0 \\ a\neq 1. \end{cases}$

Let us solve Inequality (22) in each of these cases:

(1) For $a=1$ Inequality (22) takes the form: $\frac{2x+2}{x-2}>0$, whence we find: $x<-1$; $x>2$.

(2) For $a = 0$ Inequality (22) has no solution.

(3) If $\begin{cases} a \neq 0 \\ a \neq 1, \end{cases}$ then, factoring the numerator of the left-hand side of Inequality (22), we get the inequality

$$\frac{(1-a)(x+1)\left(x-\frac{a+1}{a-1}\right)}{x-\frac{2}{a}} > 0 \qquad (23)$$

which is equivalent to Inequality (22), and, hence, to Inequality (21).

Inequality (23), in turn, must be considered in two cases: $\begin{cases} a \neq 0 \\ a < 1 \end{cases}$ and $a > 1$.

In the first case $1 - a > 0$ and Inequality (23) takes the form:

$$\frac{(x+1)\left(x-\frac{a+1}{a-1}\right)}{x-\frac{2}{a}} > 0, \qquad (24)$$

in the second case $1 - a < 0$ and Inequality (23) takes the form:

$$\frac{(x+1)\left(x-\frac{a+1}{a-1}\right)}{x-\frac{2}{a}} < 0. \qquad (25)$$

To solve Inequalities (24) and (25) by the methods of intervals, it is necessary to arrange the points -1, $\frac{a+1}{a-1}$, $\frac{2}{a}$ on the number line in the increasing order. To this end, we form the following differences:

$$A_1 = \frac{a+1}{a-1} - (-1), \quad A_2 = \frac{2}{a} - (-1), \quad A_3 = \frac{a+1}{a-1} - \frac{2}{a}$$

and find the sign of each of them.

Consider the difference $A_1 = \frac{2a}{a-1}$.

From Fig. 31 we obtain: if $a < 0$, then $A_1 > 0$; if $0 < a < 1$, then $A_1 < 0$; if $a > 1$, then $A_1 > 0$.

Analysing the difference $A_2 = \frac{2+a}{a}$, we get (Fig. 32): if $a < -2$, then $A_2 > 0$; if $-2 < a < 0$, then $A_2 < 0$; if $0 < a < 1$; $a > 1$, then $A_2 > 0$; finally, if $a = -2$, then $A_2 = 0$.

Let us now consider the difference

$$A_3 = \frac{a^2 - a + 2}{a(a-1)}.$$

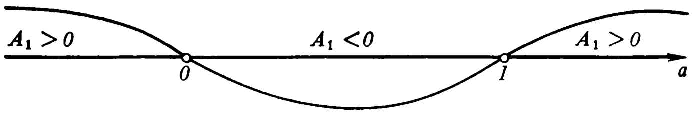

Fig. 31

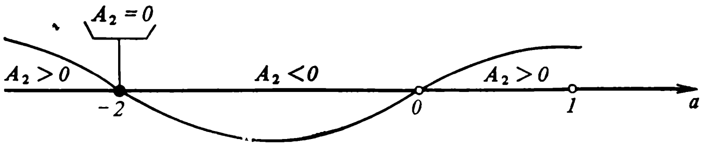

Fig. 32

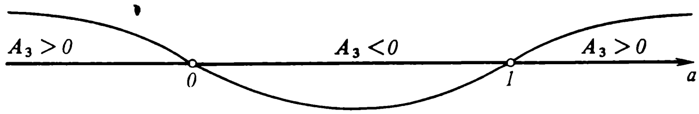

Fig. 33

Since the discriminant of the quadratic trinomial $a^2 - a + 2$ is negative, and the coefficient of a^2 is positive, $a^2 - a + 2 > 0$ for any values of a, and the sign of the difference A_3 depends only on the sign of the denominator $a(a - 1)$. We obtain (Fig. 33) that if

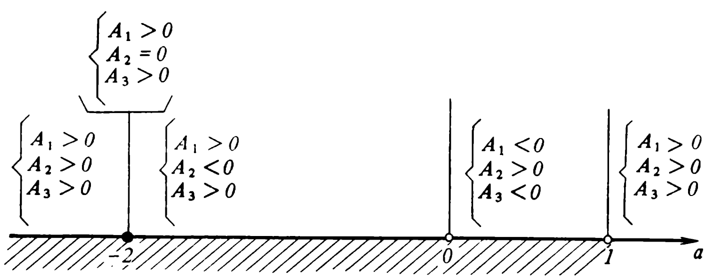

Fig. 34

$a < 0$, then $A_3 > 0$; if $0 < a < 1$, then $A_3 < 0$; if $a > 1$, then $A_3 > 0$.

Let us now illustrate the results of investigating the signs of the differences A_1, A_2, A_3 (Fig. 34). Inequality (24) is solved on the condition that $0 \neq a < 1$ (in Fig. 34 these values of a are hatched), therefore this inequality must be considered in each of the following cases: $a < -2$; $-2 < a < 0$; $0 < a < 1$; $a = -2$.

In the first three cases we get, respectively:

$$-1<\frac{2}{a}<\frac{a+1}{a-1}; \quad \frac{2}{a}<-1<\frac{a+1}{a-1}; \quad \frac{a+1}{a-1}<-1<\frac{2}{a}.$$

Solving Inequality (24) by the method of intervals (Fig. 35), we find:

if $a<-2$, then $-1<x<\frac{2}{a}$; $x>\frac{a+1}{a-1}$;

if $-2<a<0$, then $\frac{2}{a}<x<-1$; $x>\frac{a+1}{a-1}$;

if $0<a<1$, then $\frac{a+1}{a-1}<x<-1$; $x>\frac{2}{a}$.

Finally, for $a=-2$ Inequality (24) takes the form:

$$\frac{(x+1)\left(x+\frac{1}{3}\right)}{x+1}>0,$$

whence we find $x>-\frac{1}{3}$.

When solving Inequality (25), we are interested in the signs of the differences A_1, A_2, A_3 only in the interval $a>1$ (this interval

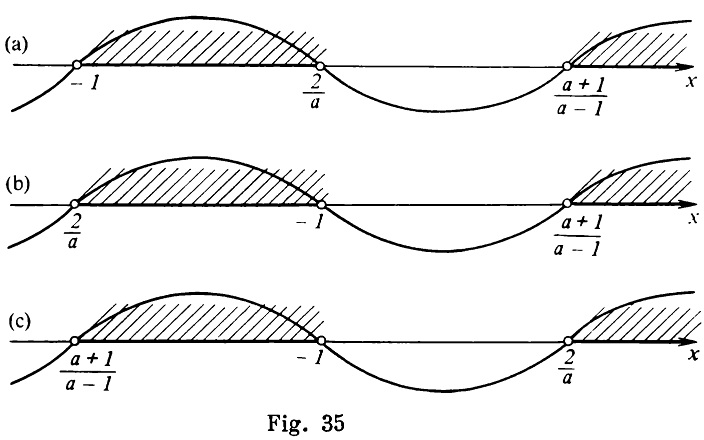

Fig. 35

is not hatched in Fig. 34). Hence, for $a>1$ we have:

$$-1<\frac{2}{a}<\frac{a+1}{a-1}.$$

Applying the method of intervals, we find the solution of Inequality (25):

if $a>1$, then $x<-1$; $\frac{2}{a}<x<\frac{a+1}{a-1}$.

Let us now write the final answer for Inequality (21):

(1) if $a<-2$, then $-1<x<\frac{2}{a}$; $x>\frac{a+1}{a-1}$;

(2) if $a=-2$, then $x>-\frac{1}{3}$;

(3) if $-2<a<0$, then $\frac{2}{a}<x<-1$; $x>\frac{a+1}{a-1}$;

(4) if $a=0$, then the inequality has no solution;

(5) if $0<a<1$, then $\frac{a+1}{a-1}<x<-1$; $x>\frac{2}{a}$;

(6) if $a=1$, then $x<-1$; $x>2$;

(7) if $a>1$, then $x<-1$; $\frac{2}{a}<x<\frac{a+1}{a-1}$.

Example 9. Find all values of the parameter a for which the system of equations

$$\begin{cases} -4x+ay=1+a \\ (6+a)x+2y=3+a \end{cases} \tag{26}$$

has no solution.

Solution. The given system is incompatible if and only if

$$\frac{-4}{6+a}=\frac{a}{2}\neq\frac{1+a}{3+a}. \tag{27}$$

From the equation $\frac{-4}{6+a}=\frac{a}{2}$ we find: $a_1=-2$; $a_2=-4$.

From the equation $\frac{a}{2}=\frac{1+a}{3+a}$ we find: $a_3=1$; $a_4=-2$.

Hence, the condition $\frac{a}{2}\neq\frac{1+a}{3+a}$ is fulfilled if $a\neq 1$; $a\neq-2$.

From the system $\begin{cases} a=-2;\ a=-4 \\ a\neq 1 \\ a\neq -2 \end{cases}$ we find that Condition (27) is equivalent to the equality $a=-4$. Thus, System (26) has no solution for $a=-4$.

Example 10. Find all values of the parameter a for which the inequality $(x-2+3a)(x-2a+3)<0$ is fulfilled for all x's belonging to $[2, 3]$.

Solution. The given inequality has the form $(x-x_1)(x-x_2)<0$, where $x_1=2-3a$, $x_2=2a-3$. Solving it, we get: $x_1<x<x_2$ (if $x_1<x_2$) or $x_2<x<x_1$ (if $x_2<x_1$); if $x_1=x_2$, then there is no solution.

Thus, the solution of the given inequality is either the interval $(2a-3, 2-3a)$ or the interval $(2-3a, 2a-3)$ (Fig. 36).

From the conditions of the problem it follows that all the points from the interval [2, 3] must satisfy the given inequality, and this is fulfilled if and only if the points with the coordinates 2 and 3 lie

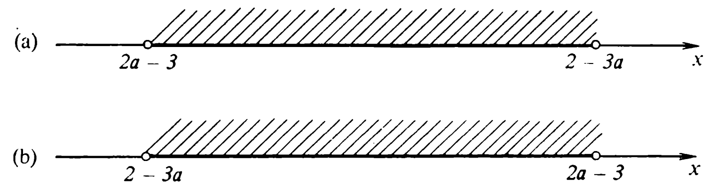

Fig. 36

inside either the interval (x_1, x_2) or (x_2, x_1), that is, if $2a - 3 < 2 < 3 < 2 - 3a$ or if $2 - 3a < 2 < 3 < 2a - 3$.

From the system of inequalities $2a - 3 < 2 < 3 < 2 - 3a$ we get the system: $\begin{cases} 2a - 3 < 2 \\ 2 - 3a > 3, \end{cases}$ whence we find: $a < -\frac{1}{3}$.

The system of inequalities $2 - 3a < 2 < 3 < 2a - 3$ is equivalent to the system: $\begin{cases} 2 - 3a < 2 \\ 2a - 3 > 3, \end{cases}$ whence we find: $a > 3$. Thus, the given inequality is fulfilled for all $x \in [2, 3]$ for $a < -\frac{1}{3}$ or $a > 3$.

Example 11. Find all values of the parameter a for which the equation

$$x^2 + 4x - 2\,|x - a| + 2 - a = 0 \tag{28}$$

has two roots.

Solution. The given equation is equivalent to the collection of two mixed systems

$$\left[\begin{array}{l} \begin{cases} x - a \geqslant 0 \\ x^2 + 4x - 2(x - a) + 2 - a = 0 \end{cases} \quad (29) \\ \begin{cases} x - a \leqslant 0 \\ x^2 + 4x + 2(x - a) + 2 - a = 0. \end{cases} \quad (30) \end{array}\right.$$

Solving System (29), we have: $\begin{cases} x \geqslant a \\ x^2 + 2x + a + 2 = 0. \end{cases}$

The discriminant D of the equation $x^2 + 2x + a + 2 = 0$ is equal to $(-a - 1)$. If $D < 0$, that is, $a > -1$, then the equation $x^2 + 2x + a + 2 = 0$ has no root; if $D = 0$, that is, $a = -1$, then this equation has the only root $x = -1$; if $D > 0$, that is, $a < -1$, then the equation has two roots: $x_1 = -1 - \sqrt{-a - 1}$, $x_2 = -1 + \sqrt{-a - 1}$.

The found roots must satisfy the inequality $x \geqslant a$: only in this case they may be regarded as solutions of the mixed system (29). We have to consider two cases: (1) $a = -1$; (2) $a < -1$ (for $a > -1$ the equation of System (29), as was noted above, has no root, hence, System (29) also has no solution).

(1) If $a = -1$, then $x = -1$. In this case, the inequality $x \geqslant a$ is fulfilled, hence, $x = -1$ is the solution of System (29).

(2) If $a < -1$, then $x_1 = -1 - \sqrt{-a-1}$, $x_2 = -1 + \sqrt{-a-1}$.

Let us find for what values of a the inequality $x_1 \geqslant a$ is fulfilled and for what values of a the inequality $x_2 \geqslant a$ is fulfilled. We begin with the inequality $x_1 \geqslant a$.

We have in succession:

$$\begin{aligned} &-1 - \sqrt{-a-1} \geqslant a, \\ &\sqrt{-a-1} \leqslant -a-1. \end{aligned} \tag{31}$$

Dividing both sides of Inequality (31) by the expression $\sqrt{-a-1}$ which takes on only positive values for $a < -1$, we get the inequality $1 \leqslant \sqrt{-a-1}$, equivalent to Inequality (31).

We then have: $1 \leqslant -a-1$, whence $a \leqslant -2$.

Let us now consider the inequality $x_2 \geqslant a$. We have:

$$-1 + \sqrt{-a-1} \geqslant a, \qquad \sqrt{-a-1} \geqslant a+1.$$

Since for $a < -1$ the left-hand side of this inequality is positive, and the right-hand side is negative, the inequality is true for all $a < -1$.

Finally, we get the following solutions of System (29): if $a > -1$, then there is no solution; if $a = -1$, then $x = -1$; if $-2 < a < -1$, then $x = -1 + \sqrt{-a-1}$; if $a \leqslant -2$, then $x_1 = -1 - \sqrt{-a-1}$, $x_2 = -1 + \sqrt{-a-1}$.

Solving System (30), we have: $\begin{cases} x \leqslant a \\ x^2 + 6x + 2 - 3a = 0. \end{cases}$

From the equation $x^2 + 6x + 2 - 3a = 0$ we find:

$$x_{3,4} = -3 \pm \sqrt{7+3a}.$$

If $a < -\frac{7}{3}$, then there is no real root, hence, System (30) has no solution; if $a = -\frac{7}{3}$, then $x = -3$; if $a > -\frac{7}{3}$, then $x_3 = -3 - \sqrt{7+3a}$, $x_4 = -3 + \sqrt{7+3a}$.

From the found roots we choose those which satisfy the inequality $x \leqslant a$. If $a = -\frac{7}{3}$, then $x = -3$ and the inequality $x \leqslant a$ is fulfilled. Hence, $x = -3$ is the solution of System (30).

Let $a > -\frac{7}{3}$ and let us find for what values of a the inequality $x_3 \leqslant a$ is fulfilled. We have:

$$-3 - \sqrt{7 + 3a} \leqslant a,$$

$$\sqrt{7 + 3a} \geqslant -a - 3. \qquad (32)$$

Since for $a > -\frac{7}{3}$ the left-hand side of Inequality (32) is positive, and the right-hand side is negative, Inequality (32) is true.

Hence, x_3 is the solution of System (30) for all $a > -\frac{7}{3}$.

Let us now consider the inequality $x_4 \leqslant a$. We have:

$$-3 + \sqrt{7 + 3a} \leqslant a,$$

$$\sqrt{7 + 3a} \leqslant a + 3. \qquad (33)$$

Since for $a > -\frac{7}{3}$ both sides of Inequality (33) are positive, squaring them, we get an equivalent inequality: $7 + 3a \leqslant (a + 3)^2$. Further, we have: $(a + 1)(a + 2) \geqslant 0$, whence we find: $a \leqslant -2$ or $a \geqslant -1$. Thus, x_4 is a solution of System (30) if $-\frac{7}{3} < a \leqslant -2$ or $a \geqslant -1$.

Finally, we get the following solutions of System (30):

if $a < -\frac{7}{3}$, then there is no solution; if $a = -\frac{7}{3}$, then $x = -3$;

if $-\frac{7}{3} < a \leqslant -2$, then $x_{3,4} = -3 \pm \sqrt{7 + 3a}$;

if $-2 < a < -1$, then $x = x_3 = -3 - \sqrt{7 + 3a}$;

if $a \geqslant -1$, then $x_{3,4} = -3 \pm \sqrt{7 + 3a}$.

We have found the solutions of Systems (29) and (30). The solution of Equation (29) is the union of the solutions found for Systems (29) and (30).

From the aforegoing reasoning it is clear that this union should be formed separately for the following values of the parameter: (1) $a > -1$; (2) $a = -1$; (3) $-2 < a < -1$; (4) $a = -2$; (5) $-\frac{7}{3} < a < -2$; (6) $a = -\frac{7}{3}$; (7) $a < -\frac{7}{3}$.

(1) If $a > -1$, then the equation has two roots: x_3, x_4, that is, $-3 \pm \sqrt{7 + 3a}$.

(2) If $a = -1$, then the equation has two roots: -1, -5.

(3) If $-2 < a < -1$, then the equation has two roots: x_2, x_3, i.e. $-1 + \sqrt{-a - 1}$ and $-3 - \sqrt{7 + 3a}$.

(4) If $a = -2$, then the equation has three roots: -2, 0, -4.

(5) If $-\frac{7}{3} < a < -2$, then the equation has four roots: $-1 \pm \sqrt{-a-1}$; $-3 \pm \sqrt{7+3a}$.

(6) If $a = -\frac{7}{3}$, then the equation has three roots: -3, $-1 \pm \frac{2}{\sqrt{3}}$.

(7) If $a < -\frac{7}{3}$, then the equation has two roots: $x_1 = -1 + \sqrt{-a-1}$, $x_2 = -1 - \sqrt{-a-1}$.

Thus, Equation (28) has two roots for $a > -2$ or for $a < -\frac{7}{3}$.

Example 12. Find all values of a for which the equation

$$2 \log (x + 3) = \log ax \tag{34}$$

has the only root.

Solution. We transform the equation to the form $\log (x + 3)^2 = \log ax$.

Then we get: $(x + 3)^2 = ax$, whence

$$x^2 - (a - 6)\, x + 9 = 0. \tag{35}$$

Equation (34) has the only root in the following cases: (1) Equation (35) has the only root and this root satisfies Equation (34); (2) Equation (35) has two roots, but one of them is extraneous for Equation (34).

Consider the first case. Equation (35) has one root if its discriminant D is equal to zero. We have: $D = (a - 6)^2 - 36 = a^2 - 12a$.

$D = 0$ for $a = 0$ or for $a = 12$. The case when $a = 0$ drops out since for $a = 0$ the right-hand side of Equation (34) is not defined. If $a = 12$, then we find from Equation (35): $x = 3$ which is the only root of Equation (35) and which, as a check shows, also satisfies Equation (34).

Consider the second case when $D > 0$. In this case Equation (35) has two roots: $x_{1,2} = \frac{a-6 \pm \sqrt{a^2-12a}}{2}$.

In order for the found roots to be the roots of Equation (34), it is necessary and sufficient that they satisfy the inequality $x + 3 > 0$. Hence, one of the found roots of Equation (35) will be a root of Equation (34), and the other will not if and only if

$$\begin{cases} x_1 > -3 \\ x_2 \leqslant -3 \end{cases} \text{ or } \begin{cases} x_2 > -3 \\ x_1 \leqslant -3, \end{cases}$$

where $x_1 = \frac{a-6+\sqrt{a^2-12a}}{2}$, $x_2 = \frac{a-6-\sqrt{a^2-12a}}{2}$.

Thus, the problem is reduced to solving the collection of two systems of inequalities:

$$\begin{cases} \dfrac{a-6+\sqrt{a^2-12a}}{2} > -3 \\ \dfrac{a-6-\sqrt{a^2-12a}}{2} \leqslant -3 \end{cases}; \quad \begin{cases} \dfrac{a-6-\sqrt{a^2-12a}}{2} > -3 \\ \dfrac{a-6+\sqrt{a^2-12a}}{2} \leqslant -3. \end{cases}$$

Solving the first system, we have:

$$\begin{cases} \sqrt{a^2-12a} > -a \\ \sqrt{a^2-12a} \geqslant a, \end{cases}$$

whence $a^2-12a > a^2$, i.e. $a < 0$.

Solving the second system, we have:

$$\begin{cases} \sqrt{a^2-12a} < a \\ \sqrt{a^2-12a} \leqslant -a. \end{cases}$$

This system has no solution since either $a < 0$ or $-a < 0$, that is, either the first or the second inequality of the last system has no solution. Thus, the second case occurs for $a < 0$.

The final result: Equation (34) has the only root if $a = 12$ or if $a < 0$.

EXERCISES

In Problems **1119** through **1155**, solve the given equations:

1119. $(a^2-2a+1)\,x = a^2+2a-3$. **1120.** $(a^3-a^2-4a+4)\,x = a-1$.

1121. $\dfrac{x}{a}+\dfrac{a}{3}+\dfrac{x+a}{a+3}=1$. **1122.** $\dfrac{x+a}{1+a}=\dfrac{x-a}{2+a}$.

1123. $\dfrac{3x-2}{a^2-2a}+\dfrac{x-1}{a-2}+\dfrac{2}{a}=0$. **1124.** $x^2-4ax+3a^2=0$.

1125. $ax^2-(1-2a)\,x+a-2=0$.

1126. $(2a-1)\,x^2-(3a+1)\,x+a-1=0$.

1127. $(a^2+a-2)\,x^2+(2a^2+a+3)\,x+a^2-1=0$.

1128. $\dfrac{3x^2-2}{a^2+3a}+\dfrac{x-1}{a+3}+\dfrac{2}{a}=0$.

1129. $\dfrac{x^2}{x-2a}+\dfrac{2ax-(a-1)(a+2)}{2a-x}+1=0$.

1130. $\dfrac{2x}{2x+a}+\dfrac{x}{2x-a}=\dfrac{a^2}{4x^2-a^2}$. **1131.** $\dfrac{2x-1}{x-a}+\dfrac{2x}{a}=\dfrac{ax-2}{a^2-ax}$.

1132. $\dfrac{x-a}{x-1}+\dfrac{x+a}{x+1}=\dfrac{x-2a}{x-2}+\dfrac{x+2a}{x+2}-\dfrac{6(a-1)}{5}$.

1133. $x\sqrt{3}+ax+\sqrt{x}=0$. **1134.** $\sqrt{x+a}=a-\sqrt{x}$.

1135. $x+\sqrt{x^2-x}=a$. **1136.** $\sqrt{x-2a}-\sqrt{x-a}=2$.

1137. $\sqrt{x^2+3a^2}-\sqrt{x^2-3a^2}=x\sqrt{2}$.

1138. $2\sqrt{a+x}+\sqrt{a-x}=\sqrt{a-x+\sqrt{x(a+x)}}$.

1139. $\frac{\sqrt{a+x}}{a}+\frac{\sqrt{a+x}}{x}=\sqrt{x}$.

1140. $\frac{1}{\sqrt{x+a}}+\frac{1}{\sqrt{x-a}}=\frac{1}{\sqrt{x^2-a^2}}$.

1141. $(4a-15)x^2+2a|x|+4=0$. **1142.** $\log_9 x+\log_9\frac{2-x}{2}=\log_9\log_9 a$.

1143. $144^{|x|}-2\times 12^{|x|}+a=0$. **1144.** $3\times 4^{x-2}+27=a+a\cdot 4^{x-2}$.

1145. $1-\log a=\frac{1}{3}\left(\log\frac{1}{2}+\log x+\frac{1}{3}\log a\right)$.

1146. $\log 2x+\log(2-x)=\log\log a$. **1147.** $\log_a x+\log_{\sqrt{a}} x+\log_{\sqrt[3]{a^2}} x=27$.

1148. $\log_a\sqrt{4+x}+3\log_{a^2}(4-x)-\log_{a^4}(16-x^2)^2=2$.

1149. $2-\log_{a^2}(1+x)=3\log_a\sqrt{x-1}-\log_{a^4}(x^2-1)^2$.

1150. $x^{\log_a x}=a^2x$. **1151.** $a^{2\log x-\log(6-x)}=1$.

1152. $a^{1+\log_3 x}+a^{1-\log_3 x}=a^2+1$.

1153. $\log_a(1-\sqrt{1-x})=\log_{a^2}(3-\sqrt{1+x})$.

1154. $\log_{\sqrt{x}} a\cdot\log_{a^2}\frac{a^2-4}{2a-x}=1$.

1155. $\frac{\log_x(2a-x)}{\log_x 2}+\frac{\log_a\sqrt{x}}{\log_a\sqrt{2}}=\frac{1}{\log_{a^2-1}2}$.

In Problems **1156** through **1164**, solve the given systems of equations:

1156. $\begin{cases}(3+a)x+2y=3\\ ax-y=3.\end{cases}$ **1157.** $\begin{cases}(7-a)x+ay=5\\ (1+a)x+3y=5.\end{cases}$

1158. $\begin{cases}x+ay=1\\ ax+y=a^2.\end{cases}$ **1159.** $\begin{cases}x+y=a\\ x^4+y^4=a^4.\end{cases}$

1160. $\begin{cases}(x-y)(x^2-y^2)=3a^3\\ (x+y)(x^2+y^2)=15a^3.\end{cases}$

1161. $\begin{cases}x+y+z=1\\ x+ay+z=a\\ x+y+az=a^2.\end{cases}$ **1162.** $\begin{cases}ax+y=z\\ y+z=3ax\\ y^3+z^3=9a^3x^3.\end{cases}$

1163. $\begin{cases}x-y=8a^2\\ \sqrt{x}+\sqrt{y}=4a.\end{cases}$ **1164.** $\begin{cases}\sqrt{x}-\sqrt{y}=1\\ \sqrt{x}+\sqrt{y}=a.\end{cases}$

In Problems **1165** through **1186**, solve the indicated inequalities:

1165. $a^2+ax<1-x$. **1166.** $2x+3(ax-8)+\frac{x}{3}<4\left(x+\frac{1}{2}\right)-5$.

1167. $\frac{3ax+4}{3a+9}<\frac{x}{a+3}+\frac{3a-5}{3a-9}$.

1168. $\dfrac{2ax+3}{5x-4a}<4.$ 1169. $\left|\dfrac{ax-5}{3}+x\right|<3.$

1170. $(2.5a+1)x^2+(a+2)x+a\leqslant 0.$

1171. $\sqrt{x}+2ax+3x>0.$ 1172. $\sqrt{\dfrac{3x-1}{a-2}}<1.$

1173. $2\sqrt{x+a}>x+1.$ 1174. $\sqrt{x}-\sqrt{x-1}>a.$

1175. $\sqrt{x^2+x}<a-x.$ 1176. $\sqrt{a+x}+\sqrt{a-x}>a.$

1177. $\sqrt{1-x^2}<a-x.$ 1178. $\sqrt{a^2-x^2}+\sqrt{2ax-x^2}>a.$

1179. $\sqrt{2ax-x^2}\geqslant a-x.$ 1180. $\log_a(x-1)+\log_a x>2.$

1181. $\log_{\frac{1}{2}}(x^2-2x+a)>-3.$ 1182. $\log_x(x-a)>2.$

1183. $\dfrac{\log_a(35-x^3)}{\log_a(5-x)}>3.$ 1184. $\log_a\dfrac{1+\log_a^2 x}{1-\log_a x}<0.$

1185. $1+\log_x\dfrac{4-x}{10}<(\log\log a-1)\log_x 10.$

1186. $\log_{\sqrt{2a}}(a+2x-x^2)<2.$

1187. For what values of a do both roots of the equation $x^2-6ax+(2-2a+9a^2)=0$ exceed 3?

1188. For what values of a do both roots of the equation $x^2-ax+2=0$ belong to the interval $[0, 3]$?

1189. For what values of a does the inequality $4^x-a\cdot 2^x-a+3\leqslant 0$ have at least one solution?

1190. For what values of a is the inequality $\dfrac{x-2a-1}{x-a}<0$ fulfilled for all x's belonging to the interval $[1, 2]$?

1191. For what values of a is the inequality $(x-3a)(x-a-3)<0$ fulfilled for all x's belonging to the interval $[1, 3]$?

1192. For what values of a does the equation $x\,|x+2a|+1-a=0$ have only one root?

1193. For what values of a does the equation $x\,|x-2a|-1-a=0$ have only one root?

1194. For what values of a does the equation $x^2-4x-2\,|x-a|+a+2=0$ have two roots?

1195. For what values of a does the system

$$\begin{cases} x^2+(5a+2)x+4a^2+2a<0 \\ x^2+a^2=4 \end{cases}$$

have at least one solution?

1196. For what values of a does the system

$$\begin{cases} x^2+(2-3a)x+2a^2-2a<0 \\ ax=1 \end{cases}$$

have at least one solution?

1197. For what values of a does the system

$$\begin{cases} x^2-(3a+1)x+2a^2+2a<0 \\ x+a^2=0, \end{cases}$$

have no solution?

1198. For what values of a does the system

$$\begin{cases} x^2+\left(1-\frac{3}{2}a\right)x+\frac{a^2}{2}-\frac{a}{2}<0 \\ x=a^2-\frac{1}{2} \end{cases}$$

have no solution?

1199. For what values of a does the system

$$\begin{cases} |x^2-7x+6|+x^2+5x+6-12|x|=0 \\ x^2-2(a-2)x+a(a-4)=0 \end{cases}$$

have two solutions?

1200. For what values of a does the system

$$\begin{cases} |x^2+5x+4|-9x^2+5x+4-10x|x|=0 \\ x^2-2(a+1)x+a(a+2)=0 \end{cases}$$

have only one solution?

1201. For what values of a does the system

$$\begin{cases} |x^2+7x+6|+x^2-5x+6-12|x|=0 \\ x^2-2(a+2)x+a(a+4)=0 \end{cases}$$

have two solutions?

1202. For what values of a does the equation $\log(x^2+2ax)-\log(8x-6a-3)=0$ have the only root?

Part II

TRIGONOMETRY

Chapter 3

IDENTICAL TRANSFORMATIONS

SEC. 21. IDENTICAL TRANSFORMATIONS OF TRIGONOMETRIC FUNCTIONS

Let us recall the fundamentals of trigonometry.

I. *Some values of trigonometric functions*:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	—	0	—
$\cot x$	—	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	—	0

II. *Signs of trigonometric functions*:

Quarter	$\sin x$	$\cos x$	$\tan x$	$\cot x$
I	+	+	+	+
II	+	−	−	−
III	−	−	+	+
IV	−	+	−	−

III. *Parity. Periodicity.*

The function $y = \cos x$ is even, all the rest of trigonometric functions being odd. Thus,

$$\cos(-x) = \cos x,$$
$$\sin(-x) = -\sin x,$$
$$\tan(-x) = -\tan x \quad \left(x \neq \frac{\pi}{2} + \pi n\right),$$
$$\cot(-x) = -\cot x \quad (x \neq \pi n).^*$$

All trigonometric functions are periodic: $T = 2\pi$ is a period of the functions $y = \sin x$, $y = \cos x$, while $T = \pi$ is a period of the functions $y = \tan x$, $y = \cot x$ (we recall here that a period of a function $f(x)$ is defined as a smallest positive number p for which $f(x + p) = f(x)$). Thus,

$$\sin(x + 2\pi) = \sin(x - 2\pi) = \sin x,$$
$$\cos(x + 2\pi) = \cos(x - 2\pi) = \cos x,$$
$$\tan(x + \pi) = \tan(x - \pi) = \tan x \quad \left(x \neq \frac{\pi}{2} + \pi n\right),$$
$$\cot(x + \pi) = \cot(x - \pi) = \cot x \quad (x \neq \pi n).$$

IV. *Formulas relating trigonometric functions of the same argument (the Pythagorean identities)*:

$$\cos^2\alpha + \sin^2\alpha = 1, \tag{IV.1}$$

$$1 + \tan^2\alpha = \frac{1}{\cos^2\alpha} \quad \left(\alpha \neq \frac{\pi}{2} + \pi n\right), \tag{IV.2}$$

$$1 + \cot^2\alpha = \frac{1}{\sin^2\alpha} \quad (\alpha \neq \pi n), \tag{IV.3}$$

V. *Formulas relating trigonometric functions of two arguments one of which is twice the other (the double-angle formulas (identities))*:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha, \tag{V.1}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha, \tag{V.2}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \quad \left(\alpha \neq \frac{\pi}{4} + \frac{\pi n}{2},\ \alpha \neq \frac{\pi}{2} + \pi k\right), \tag{V.3}$$

$$\cot 2\alpha = \frac{\cot^2\alpha - 1}{2\cot\alpha} \quad \left(\alpha \neq \frac{\pi n}{2}\right), \tag{V.4}$$

$$1 + \cos 2\alpha = 2\cos^2\alpha, \tag{V.5}$$

$$1 - \cos 2\alpha = 2\sin^2\alpha, \tag{V.6}$$

$$1 \pm \sin 2\alpha = (\cos\alpha \pm \sin\alpha)^2. \tag{V.7}$$

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* In the following, if it is not specially stipulated, it is meant that $n, k, l, m, \ldots$ take on any integer values.

VI. *Addition and subtraction formulas (identities)*:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha, \quad \text{(VI.1)}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta, \quad \text{(VI.2)}$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \quad \left(\alpha \neq \frac{\pi}{2} + \pi n,\ \beta \neq \frac{\pi}{2} + \pi k,\ \alpha \pm \beta \neq \frac{\pi}{2} + \pi m\right), \quad \text{(VI.3)}$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha} \quad (\alpha \neq \pi n,\ \beta \neq \pi k,\ \alpha \pm \beta \neq \pi m). \quad \text{(VI.4)}$$

VII. *Reduction formulas*:

x	$\frac{\pi}{2}-\alpha$	$\frac{\pi}{2}+\alpha$	$\pi-\alpha$	$\pi+\alpha$	$\frac{3\pi}{2}-\alpha$	$\frac{3\pi}{2}+\alpha$	$2\pi-\alpha$
$\sin x$	$\cos\alpha$	$\cos\alpha$	$\sin\alpha$	$-\sin\alpha$	$-\cos\alpha$	$-\cos\alpha$	$-\sin\alpha$
$\cos x$	$\sin\alpha$	$-\sin\alpha$	$-\cos\alpha$	$-\cos\alpha$	$-\sin\alpha$	$\sin\alpha$	$\cos\alpha$
$\tan x$	$\cot\alpha$	$-\cot\alpha$	$-\tan\alpha$	$\tan\alpha$	$\cot\alpha$	$-\cot\alpha$	$-\tan\alpha$
$\cot x$	$\tan\alpha$	$-\tan\alpha$	$-\cot\alpha$	$\cot\alpha$	$\tan\alpha$	$-\tan\alpha$	$-\cot\alpha$

To make easier the memorizing of the reduction formulas the following mnemonic rule is recommended to be used:

(1) determine the name of a function (if the arc α is laid off from the horizontal diameter ($\pi \pm \alpha$, $2\pi - \alpha$), then the name of the function is retained, and if the arc α is laid off from the vertical diameter $\left(\frac{\pi}{2} \pm \alpha, \frac{3\pi}{2} \pm \alpha\right)$, then sine, cosine, tangent, cotangent are changed into cosine, sine, cotangent, tangent, respectively);

(2) determine the sign of a function: regarding the arc α as a first-quadrant arc, find the quadrant in which the arc $\frac{\pi n}{2} \pm \alpha$ is situated and determine the sign of the given function in this quadrant.

VIII. *Formulas for transforming a sum of trigonometric functions into a product (the sum formulas (identities))*:

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}, \quad \text{(VIII.1)}$$

$$\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}, \quad \text{(VIII.2)}$$

$$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}, \qquad \text{(VIII.3)}$$

$$\cos\alpha-\cos\beta=2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}, \qquad \text{(VIII.4)}$$

$$\tan\alpha\pm\tan\beta=\frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}\ \left(\alpha\neq\frac{\pi}{2}+\pi n,\ \beta\neq\frac{\pi}{2}+\pi k\right), \qquad \text{(VIII.5)}$$

$$\cot\alpha\pm\cot\beta=\frac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta}\ (\alpha\neq\pi n,\ \beta\neq\pi k), \qquad \text{(VIII.6)}$$

$$\cos\alpha\pm\sin\alpha=\sqrt{2}\cos\left(\frac{\pi}{4}\mp\alpha\right). \qquad \text{(VIII.7)}$$

IX. *Formulas for transforming a product of trigonometric functions into a sum (the product formulas (identities)):*

$$\sin\alpha\cos\beta=\frac{\sin(\alpha-\beta)+\sin(\alpha+\beta)}{2}, \qquad \text{(IX.1)}$$

$$\cos\alpha\cos\beta=\frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}, \qquad \text{(IX.2)}$$

$$\sin\alpha\sin\beta=\frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}. \qquad \text{(IX.3)}$$

Example 1. Simplify the function

$$f(\alpha)=\frac{\sin^3(\alpha-270^\circ)\cos(360^\circ-\alpha)}{\tan^3(\alpha-90^\circ)\cos^3(\alpha-270^\circ)}.$$

Solution. Taking advantage of the fact that the function $y=\cos x$ is even and the functions $y=\sin x$, $y=\tan x$ are odd, we get:

$$f(\alpha)=\frac{-\sin^3(270^\circ-\alpha)\cos(360^\circ-\alpha)}{-\tan^3(90^\circ-\alpha)\cos^3(270^\circ-\alpha)}.$$

Applying the reduction formulas, we get:

$$f(\alpha)=\frac{-\cos^3\alpha\cdot\cos\alpha}{\cot^3\alpha\,(-\sin^3\alpha)}=\frac{\cos^4\alpha}{\dfrac{\cos^3\alpha}{\sin^3\alpha}\cdot\sin^3\alpha}=\cos\alpha$$

The original function is identical to $\cos\alpha$ on the set of all such α's that $\sin\alpha\neq0$, $\cos\alpha\neq0$, that is, for $\alpha\neq\frac{\pi n}{2}$.

Example 2. Prove the identity

$$\frac{\cos^2\alpha}{\cot\dfrac{\alpha}{2}-\tan\dfrac{\alpha}{2}}=\frac{1}{4}\sin 2\alpha.$$

Solution. We transform the left-hand side of the identity:

$$\frac{\cos^2\alpha}{\cot\frac{\alpha}{2}-\tan\frac{\alpha}{2}}=\frac{\cos^2\alpha}{\dfrac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}-\dfrac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}=\frac{\cos^2\alpha}{\dfrac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}}$$

$$=\cos^2\alpha\,\frac{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}.$$

But (see Formulas (V.I) and (V.2))

$$\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=\frac{1}{2}\sin\alpha,\quad \cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}=\cos\alpha,$$

therefore

$$\cos^2\alpha\,\frac{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}-\cos^2\frac{\alpha}{2}}=\frac{1}{2}\cos^2\alpha\,\frac{\sin\alpha}{\cos\alpha}=\frac{1}{2}\sin\alpha\cos\alpha=\frac{1}{4}\sin 2\alpha.$$

This identity is true on the condition that $\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\neq 0$ and $\cos\alpha\neq 0$, that is, for $\sin\alpha\neq 0$ and $\cos\alpha\neq 0$, and, hence, for $\alpha\neq\frac{\pi n}{2}$.

Example 3. Prove the identity

$$\frac{\tan^2 2\alpha-\tan^2\alpha}{1-\tan^2 2\alpha\tan^2\alpha}=\tan 3\alpha\tan\alpha.$$

Solution. We factor both the numerator and denominator of the expression contained on the left-hand side of the identity:

$$\frac{\tan^2 2\alpha-\tan^2\alpha}{1-\tan^2 2\alpha\tan^2\alpha}=\frac{(\tan 2\alpha-\tan\alpha)(\tan 2\alpha+\tan\alpha)}{(1-\tan 2\alpha\tan\alpha)(1+\tan 2\alpha\tan\alpha)}$$

$$=\frac{\tan 2\alpha+\tan\alpha}{1-\tan 2\alpha\tan\alpha}\cdot\frac{\tan 2\alpha-\tan\alpha}{1+\tan 2\alpha\tan\alpha}.$$

Further, using Formula (VI.3), we get:

$$\tan(2\alpha+\alpha)\cdot\tan(2\alpha-\alpha)=\tan 3\alpha\cdot\tan\alpha.$$

The proved identity is true for $\alpha\neq\frac{\pi}{2}+\pi k$, $2\alpha\neq\frac{\pi}{2}+\pi n$, $3\alpha\neq\frac{\pi}{2}+\pi m$, that is, for $\alpha\neq\frac{\pi}{4}+\frac{\pi n}{2}$ and $\alpha\neq\frac{\pi}{6}+\frac{\pi m}{3}$ (the

set P of all numbers of the form $\frac{\pi}{2}+\pi k$ is contained in the set M of all numbers of the form $\frac{\pi}{6}+\frac{\pi m}{3}\Big)$.

Example 4. Prove the identity

$$4 \sin \alpha \sin (60^\circ - \alpha) \sin (60^\circ + \alpha) = \sin 3\alpha.$$

Solution. Here, it is expedient to apply the formulas from Group IX to the left-hand side of the identity. We have:

$$4 \sin \alpha \sin (60^\circ - \alpha) \sin (60^\circ + \alpha)$$

$$= 4 \sin \alpha \, \frac{\cos (60^\circ - \alpha - 60^\circ - \alpha) - \cos (60^\circ - \alpha + 60^\circ + \alpha)}{2}$$

$$= 2 \sin \alpha (\cos (-2\alpha) - \cos 120^\circ) = 2 \sin \alpha \left(\cos 2\alpha + \frac{1}{2}\right)$$

$$= 2 \sin \alpha \cos 2\alpha + \sin \alpha = 2\, \frac{\sin (\alpha - 2\alpha) + \sin (\alpha + 2\alpha)}{2} + \sin \alpha$$

$$= -\sin \alpha + \sin 3\alpha + \sin \alpha = \sin 3\alpha.$$

Thus, the identity is valid for all real values of α.

Example 5. Check the equality

$$\sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ = \cos 7^\circ.$$

Solution. We use the formulas from Group VIII for transforming the left-hand side of the identity. We have:

$$(\sin 47^\circ + \sin 61^\circ) - (\sin 11^\circ + \sin 25^\circ)$$

$$= 2 \sin 54^\circ \cos 7^\circ - 2 \sin 18^\circ \cos 7^\circ = 2 \cos 7^\circ (\sin 54^\circ - \sin 18^\circ)$$

$$= 2 \cos 7^\circ \cdot 2 \sin 18^\circ \cos 36^\circ.$$

If the obtained expression is multiplied or divided by $\cos 18^\circ$, we can apply the formula $2 \sin 18^\circ \cos 18^\circ = \sin 36^\circ$ and get:

$$2 \cos 7^\circ \cdot \frac{\sin 36^\circ \cos 36^\circ}{\cos 18^\circ} = \cos 7^\circ \cdot \frac{\sin 72^\circ}{\cos 18^\circ} = \cos 7^\circ \cdot \frac{\cos 18^\circ}{\cos 18^\circ} = \cos 7^\circ.$$

Hence, the original equality is true.

Remark. In many cases when there is a product of the form $\sin \alpha \times \cos 2\alpha \cos 4\alpha \cdot \ldots \cdot \cos 2^n\alpha$ or of the form $\cos \alpha \cos 2\alpha \cos 4\alpha \times \ldots \cdot \cos 2^n\alpha$, the method used in Example 5 turns out to be useful. According to this method, the given expression is multiplied and divided by either $\cos \alpha$ or $\sin \alpha$. Then we use the formula $2 \sin \alpha \times \cos \alpha = \sin 2\alpha$, and $2 \sin 2\alpha \cos 2\alpha = \sin 4\alpha$. Let us illustrate this by an example.

Example 6. Compute

$$\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}.$$

Solution. Let us denote the given product by A, and multiply and divide it by $2\sin\frac{\pi}{65}$. Since $2\sin\frac{\pi}{65}\cos\frac{\pi}{65}=\sin\frac{2\pi}{65}$, we have:

$$A=\frac{\sin\frac{2\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{6}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}}{2\sin\frac{\pi}{65}}$$

Further we have:

$$\sin\frac{2\pi}{65}\cos\frac{2\pi}{65}=\frac{1}{2}\sin\frac{4\pi}{65};$$

$$\sin\frac{4\pi}{65}\cos\frac{4\pi}{65}=\frac{1}{2}\sin\frac{8\pi}{65}$$

and so forth. In the final analysis, we get:

$$A=\frac{\sin\frac{64\pi}{65}}{2^6\sin\frac{\pi}{65}}=\frac{\sin\left(\pi-\frac{\pi}{65}\right)}{64\sin\frac{\pi}{65}}=\frac{\sin\frac{\pi}{65}}{64\sin\frac{\pi}{65}}=\frac{1}{64}.$$

Example 7. It is known that $\tan\alpha=-\frac{3}{4}$ and $\frac{\pi}{2}<\alpha<\pi$. Find the values of the remaining trigonometric functions of the argument α.

Solution. First of all we find the value of $\cot\alpha$.

We have: $\cot\alpha=\frac{1}{\tan\alpha}=-\frac{4}{3}$. Then from Formula (IV.2) we get:

$$\cos^2\alpha=\frac{1}{1+\tan^2\alpha}=\frac{16}{25}.$$

Hence, $\cos\alpha=\frac{4}{5}$ or $\cos\alpha=-\frac{4}{5}$. But, by the hypothesis, α belongs to Quadrant II, where cosine takes on only negative values. Thus, $\cos\alpha=-\frac{4}{5}$.

Since $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$, $\sin\alpha=\tan\alpha\cos\alpha$, whence $\sin\alpha=\frac{3}{5}$. Thus, $\cot\alpha=-\frac{4}{3}$, $\cos\alpha=-\frac{4}{5}$, $\sin\alpha=\frac{3}{5}$.

Example 8. Compute $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\cot\alpha$ if $\alpha=112°30'$.

Solution. From Formula (V.5) it follows:

$$|\cos\alpha| = \sqrt{\frac{1+\cos 2\alpha}{2}}.$$

Since $90° < 112°30' < 180°$, we have: $\cos\alpha < 0$. By hypothesis, $2\alpha = 225°$, hence,

$$\cos 112°30' = -\sqrt{\frac{1+\cos 225°}{2}} = -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = -\frac{\sqrt{2-\sqrt{2}}}{2}.$$

Similarly, using formula (V.6) and bearing in mind that, by hypothesis, α belongs to the second quadrant, we get:

$$\sin 112°30' = \sqrt{\frac{1-\cos 225°}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2},$$

$$\tan 112°30' = -\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = -(1+\sqrt{2}),$$

$$\cot 112°30' = \frac{1}{-(1+\sqrt{2})} = 1-\sqrt{2}.$$

Example 9. Compute $\tan\frac{\alpha}{4}$ if $\cos\alpha = -0.6$ and $180° < \alpha < 270°$.

Solution. It follows from the conditions of the problem that $45° < \frac{\alpha}{4} < 67°30'$. But then $\tan\frac{\alpha}{4} > 0$. Applying Formulas (V.5) and (V.6), we get:

$$\tan\frac{\alpha}{4} = \sqrt{\frac{1-\cos\frac{\alpha}{2}}{1+\cos\frac{\alpha}{2}}}.$$

Since, by hypothesis, $180° < \alpha < 270°$, that is, $90° < \frac{\alpha}{2} < 135°$, we have: $\cos\frac{\alpha}{2} < 0$. Hence,

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1-0.6}{2}} = -\frac{\sqrt{5}}{5}$$

and

$$\tan\frac{\alpha}{4} = \sqrt{\frac{1+\frac{\sqrt{5}}{5}}{1-\frac{\sqrt{5}}{5}}} = \frac{1+\sqrt{5}}{2}.$$

Example 10. Compute $16 \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2}$ if $\cos \alpha = \frac{3}{4}$.

Solution. By Formulas (IX.3) and then (V.5) we get:

$$16 \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2} = 16 \frac{\cos\left(\frac{\alpha}{2} - \frac{3\alpha}{2}\right) - \cos\left(\frac{\alpha}{2} + \frac{3\alpha}{2}\right)}{2}$$

$$= 8 (\cos \alpha - \cos 2\alpha) = 8 (\cos \alpha - 2 \cos^2 \alpha + 1).$$

But $\cos \alpha = \frac{3}{4}$, therefore,

$$16 \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2} = 8 \left(\frac{3}{4} - 2\left(\frac{3}{4}\right)^2 + 1\right) = 5.$$

Example 11. Prove that if $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\alpha + \beta + \gamma = \frac{\pi}{2}$, then

$$\tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha = 1. \qquad (1)$$

Solution. We transform the left-hand side of the equality, taking into consideration that, by hypothesis, $\gamma = \frac{\pi}{2} - (\alpha + \beta)$:

$$\tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha$$
$$= \tan \alpha \tan \beta + \tan \gamma (\tan \beta + \tan \alpha)$$
$$= \tan \alpha \tan \beta + \tan\left(\frac{\pi}{2} - (\alpha + \beta)\right) (\tan \alpha + \tan \beta)$$
$$= \tan \alpha \tan \beta + \cot (\alpha + \beta) (\tan \alpha + \tan \beta)$$
$$= \tan \alpha \tan \beta + \frac{1}{\tan (\alpha + \beta)} (\tan \alpha + \tan \beta)$$
$$= \tan \alpha \tan \beta + \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta} (\tan \alpha + \tan \beta)$$
$$= \tan \alpha \tan \beta + 1 - \tan \alpha \tan \beta = 1.$$

Thus, identity (1) has been proved.

Example 12. Prove that if $\frac{3\pi}{4} < \alpha < \pi$, then

$$\sqrt{2 \cot \alpha + \frac{1}{\sin^2 \alpha}} = -1 - \cot \alpha. \qquad (2)$$

Proof. We have:

$$\sqrt{2 \cot \alpha + \frac{1}{\sin^2 \alpha}} = \sqrt{2 \cot \alpha + 1 + \cot^2 \alpha} = \sqrt{(1 + \cot \alpha)^2}$$
$$= | 1 + \cot \alpha |.$$

Since the inequality $\cot\alpha < -1$ is fulfilled in the interval $\frac{3\pi}{4} < \alpha < \pi$, we have in this interval: $1+\cot\alpha < 0$, and, consequently, $|1+\cot\alpha| = -1-\cot\alpha$.

Thus, if $\frac{3\pi}{4} < \alpha < \pi$, then Identity (2) has been proved.

Example 13. Prove that if $\sin\alpha + \sin\beta = 2\sin(\alpha+\beta)$, where $\alpha+\beta \neq \pi k$, then

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1}{3}. \tag{3}$$

Proof. Transforming both sides of the equality $\sin\alpha + \sin\beta = 2\sin(\alpha+\beta)$ by formulas VIII and VI, we get:

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 4\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\beta}{2}. \tag{4}$$

Since $\alpha+\beta \neq \pi k$, that is, $\frac{\alpha+\beta}{2} \neq \frac{\pi k}{2}$, we know that $\cos\frac{\alpha+\beta}{2} \neq 0$ and $\sin\frac{\alpha+\beta}{2} \neq 0$, and, therefore, Equality (4) implies:

$$\cos\frac{\alpha-\beta}{2} = 2\cos\frac{\alpha+\beta}{2}. \tag{5}$$

Consider the expression $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}$. We have:

$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}} = \frac{\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}}$$

(here we have used Formulas (IX.3) and (IX.2)). Taking advantage of Equality (5), we get:

$$\frac{\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2}} = \frac{2\cos\frac{\alpha+\beta}{2} - \cos\frac{\alpha+\beta}{2}}{2\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha+\beta}{2}} = \frac{1}{3}.$$

Thus, Equality (3) has been proved.

Example 14. Prove that if $\tan\alpha = \frac{1}{7}$, $\sin\beta = \frac{1}{\sqrt{10}}$,

$$0 < \alpha < \frac{\pi}{2} \text{ and } 0 < \beta < \frac{\pi}{2}, \text{ then } \alpha + 2\beta = \frac{\pi}{4}.$$

Solution. Compute $\tan(\alpha+2\beta)$. We have:

$$\tan(\alpha+2\beta)=\frac{\tan\alpha+\tan 2\beta}{1-\tan\alpha\tan 2\beta}=\frac{\frac{1}{7}+\tan 2\beta}{1-\frac{1}{7}\tan 2\beta}.$$

Now, we have to find the value $\tan 2\beta$. For this purpose, let us recall that $\sin\beta=\frac{1}{\sqrt{10}}$, $0<\beta<\frac{\pi}{2}$.

We have:

$$\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-\frac{1}{10}}=\frac{3}{\sqrt{10}},\quad \tan\beta=\frac{\sin\beta}{\cos\beta}=\frac{1}{3},$$

$$\tan 2\beta=\frac{2\tan\beta}{1-\tan^2\beta}=\frac{\frac{2}{3}}{1-\frac{1}{9}}=\frac{3}{4}.$$

Hence, $\tan(\alpha+2\beta)=\dfrac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\times\frac{3}{4}}=1.$

By hypothesis, $0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$, hence, $0<2\beta<\pi$. But $\tan 2\beta=\frac{3}{4}>0$, hence, $0<2\beta<\frac{\pi}{2}$, and therefore $0<\alpha+2\beta<\pi$. But in the interval $(0, \pi)$ the function $\tan x$ takes on the value 1 only at the point $\frac{\pi}{4}$. Hence, $\alpha+2\beta=\frac{\pi}{4}$.

EXERCISES

In Problems 1203 through 1219, simplify the given expressions:

1203. $\dfrac{2\cos\left(\frac{\pi}{2}-\alpha\right)\sin\left(\frac{\pi}{2}+\alpha\right)\tan(\pi-\alpha)}{\cot\left(\frac{\pi}{2}+\alpha\right)\sin(\pi-\alpha)}$.

1204. $\dfrac{\sin\left(\frac{3\pi}{2}+\alpha\right)\tan\left(\frac{\pi}{2}+\beta\right)}{\cos(\pi-\alpha)\cot\left(\frac{3\pi}{2}-\beta\right)}-\dfrac{\sin\left(\frac{3\pi}{2}-\beta\right)\cot\left(\frac{\pi}{2}+\alpha\right)}{\cos(2\pi-\beta)\tan(\pi-\alpha)}$.

1205. $\dfrac{\cot\frac{\alpha}{2}+\tan\frac{\alpha}{2}}{\cot\frac{\alpha}{2}-\tan\frac{\alpha}{2}}$. **1206.** $\dfrac{2}{\sin 4\alpha}-\cot 2\alpha$.

1207. $\frac{\tan^2(45^\circ+\alpha)-1}{\tan^2(45^\circ+\alpha)+1}$. 1208. $\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)\frac{1-\sin\alpha}{\cos\alpha}$.

1209. $\frac{2\cos^2\alpha-1}{2\tan\left(\frac{\pi}{4}-\alpha\right)\sin^2\left(\frac{\pi}{4}+\alpha\right)}$.

1210. $\cos^2(\alpha+\beta)+\cos^2(\alpha-\beta)-\cos 2\alpha\cos 2\beta$.

1211. $\frac{\sin\alpha+\sin 3\alpha+\sin 5\alpha}{\cos\alpha+\cos 3\alpha+\cos 5\alpha}$. 1212. $\frac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\sin 7\alpha}{\cos\alpha+\cos 3\alpha+\cos 5\alpha+\cos 7\alpha}$.

1213. $\frac{\sin\alpha+\sin 3\alpha+\sin 5\alpha+\ldots+\sin(2n-1)\alpha}{\cos\alpha+\cos 3\alpha+\cos 5\alpha+\ldots+\cos(2n-1)\alpha}$.

1214. $\frac{\sqrt{2}-\sin\alpha-\cos\alpha}{\sin\alpha-\cos\alpha}$. 1215. $\cos 4\alpha+4\cos 2\alpha+3$.

1216. $\frac{\sin 4\alpha}{1+\cos 4\alpha}\cdot\frac{\cos 2\alpha}{1+\cos 2\alpha}$. 1217. $\frac{\sin^2 2\alpha-4\sin^2\alpha}{\sin^2 2\alpha+4\sin^2\alpha-4}$.

1218. $\sin^2\left(\frac{\pi}{8}+\frac{\alpha}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{\alpha}{2}\right)$.

1219. $\frac{\sin(60^\circ+\alpha)}{4\sin\left(15^\circ+\frac{\alpha}{4}\right)\sin\left(75^\circ-\frac{\alpha}{4}\right)}$.

In Problems 1220 through 1236, check the indicated equalities:

1220. $\sin\frac{\pi}{12}\cos\frac{\pi}{12}=\frac{1}{4}$. 1221. $\tan 55^\circ-\tan 35^\circ=2\tan 20^\circ$.

1222. $8\cos 10^\circ\cos 20^\circ\cos 40^\circ=\cot 10^\circ$.

1223. $\cos\frac{\pi}{5}\cos\frac{3\pi}{5}=-\frac{1}{4}$. 1224. $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$.

1225. $\tan 30^\circ+\tan 40^\circ+\tan 50^\circ+\tan 60^\circ=\frac{8\sqrt{3}\cos 20^\circ}{3}$.

1226. $\sin 70^\circ+8\cos 20^\circ\cos 40^\circ\cos 80^\circ=2\cos^2 10^\circ$.

1227. $\frac{1-4\sin 10^\circ\sin 70^\circ}{2\sin 10^\circ}=1$.

1228. $\cos 24^\circ+\cos 48^\circ-\cos 84^\circ-\cos 12^\circ=\frac{1}{2}$.

1229. $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}=-\frac{1}{2}$.

1230. $\tan 20^\circ+\tan 40^\circ+\tan 80^\circ-\tan 60^\circ=8\sin 40^\circ$.

1231. $\tan^6 20^\circ-33\tan^4 20^\circ+27\tan^2 20^\circ=3$.

1232. $\sin^2\frac{\pi}{7}\sin^2\frac{2\pi}{7}\sin^2\frac{3\pi}{7}=\frac{7}{64}$. 1233. $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}=-\frac{1}{8}$.

1234. $\tan 55^\circ\tan 65^\circ\tan 75^\circ=\tan 85^\circ$. 1235. $\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7}=\sqrt{7}$.

1236. $\cos\frac{\pi}{20}\cos\frac{3\pi}{20}\cos\frac{7\pi}{20}\cos\frac{9\pi}{20}=-\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$.

In Problems **1237** through **1262**, prove the given identities:

1237. $\frac{\sin(\beta-\gamma)}{\cos\beta\cos\gamma}+\frac{\sin(\gamma-\alpha)}{\cos\gamma\cos\alpha}+\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=0.$

1238. $\frac{\sin^2 3\alpha}{\sin^2\alpha}-\frac{\cos^2 3\alpha}{\cos^2\alpha}=8\cos 2\alpha.$

1239. $\sqrt{\cos^2\alpha\cos^2\beta-\frac{1}{2}\sin 2\alpha\sin 2\beta+\sin^2\alpha\sin^2\beta}=|\cos(\alpha+\beta)|.$

1240. $\cot\left(\frac{3\pi}{2}-\alpha\right)\sin\left(\frac{3\pi}{2}+\alpha\right)\sin\left(\alpha-\frac{\pi}{2}\right)+\tan(\pi+\alpha)\cos(\pi+\alpha)$
$\times\cos(2\pi-\alpha)=0.$

1241. $\sin(\alpha-270°)\cos(\alpha+90°)\tan(3\alpha-180°)=\cos(180°-\alpha)$
$\times\sin(180°-\alpha)\cot(90°-3\alpha).$

1242. $3(\sin^4 x+\cos^4 x)-2(\sin^6 x+\cos^6 x)=1.$

1243. $\frac{\tan^3\alpha}{\sin^2\alpha}-\frac{1}{\sin\alpha\cos\alpha}+\frac{\cot^3\alpha}{\cos^2\alpha}=\tan^3\alpha+\cot^3\alpha.$

1244. $\frac{\cos^3\alpha-\cos 3\alpha}{\cos\alpha}+\frac{\sin^3\alpha+\sin 3\alpha}{\sin\alpha}=3.$ **1245.** $\frac{\cos 2\alpha}{1+\sin 2\alpha}=\frac{1-\tan\alpha}{1+\tan\alpha}.$

1246. $1-\sin 8\alpha=2\cos^2(45°+4\alpha).$ **1247.** $\frac{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}=\frac{1}{\cos\alpha}-\tan\alpha.$

1248. $\frac{\cot\alpha+\sin^{-1}\alpha}{\sin\alpha+\tan\alpha}=\frac{2\cos\alpha}{1-\cos 2\alpha}.$ **1249.** $\tan^2\left(45°+\frac{\alpha}{2}\right)=\frac{\cos^{-1}\alpha+\tan\alpha}{\cot^{-1}\alpha-\tan\alpha}.$

1250. $(\cos\alpha+\sin\beta)^2+(\sin\alpha-\cos\beta)^2=4\cos^2\left(45°+\frac{\alpha-\beta}{2}\right).$

1251. $2\left(\frac{1}{\sin 2\alpha}+\cot 2\alpha\right)=\cot\frac{\alpha}{2}-\tan\frac{\alpha}{2}.$ **1252.** $\frac{1-2\cos^2\varphi}{\sin\varphi\cos\varphi}=\tan\varphi-\cot\varphi.$

1253. $\sqrt{1+\sin\alpha}-\sqrt{1-\sin\alpha}=2\sin\frac{\alpha}{2}\quad\left(0<\alpha<\frac{\pi}{2}\right).$

1254. $4\sin\left(\alpha+\frac{\pi}{3}\right)\sin\left(\alpha-\frac{\pi}{3}\right)=4\sin^2\alpha-3.$

1255. $2\cos\alpha\cos\beta\cos(\alpha+\beta)=\cos^2\alpha+\cos^2\beta-\sin^2(\alpha+\beta).$

1256. $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)=4\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha+\gamma}{2}\cos\frac{\beta+\gamma}{2}.$

1257. $\frac{\sin\alpha+\sin\beta}{\tan\frac{\alpha+\beta}{2}+\cot\frac{\alpha-\beta}{2}}=\frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{2\cos\beta}.$

1258. $\cos\alpha-\frac{1}{2}\cos 3\alpha-\frac{1}{2}\cos 5\alpha=8\sin^2\alpha\cos^3\alpha.$

1259. $\frac{2\sin\alpha-\sin 3\alpha+\sin 5\alpha}{\cos\alpha-2\cos 2\alpha+\cos 3\alpha}=-\frac{2\cos 2\alpha}{\tan\frac{\alpha}{2}}.$

1260. $\cos\alpha+\cos(120°-\alpha)+\cos(120°+\alpha)=0.$

1261. $\tan(35°+\alpha)\tan(25°-\alpha)=\frac{2\cos(10°+2\alpha)-1}{2\cos(10°+2\alpha)+1}.$

1262. $$\frac{\frac{\sqrt{3}}{2}\cos 2\alpha-\frac{1}{2}\sin 2\alpha}{1-\frac{1}{2}\cos 2\alpha-\frac{\sqrt{3}}{2}\sin 2\alpha}=\tan\left(\alpha+\frac{\pi}{3}\right).$$

In Problems 1263 through 1273, compute without using tables:

1263. $\dfrac{\sin 10^\circ\cos 20^\circ+\cos 10^\circ\sin 20^\circ}{\cos 19^\circ\cos 11^\circ-\sin 19^\circ\sin 11^\circ}$. 1264. $\dfrac{\sin 9^\circ\cos 39^\circ-\cos 9^\circ\sin 39^\circ}{\cos\frac{3\pi}{7}\cos\frac{5\pi}{28}+\sin\frac{3\pi}{7}\sin\frac{5\pi}{28}}$.

1265. $\cos 15^\circ$. 1266. $\tan 15^\circ$. 1267. $\sin 285^\circ$.
1268. $\cos 165^\circ$. 1269. $\cos 292^\circ30'$.
1270. $2\sin 40^\circ+2\cos 130^\circ-3\sin 160^\circ-3\cos(-110^\circ)$.
1271. $\cos 10^\circ\cos 30^\circ\cos 50^\circ\cos 70^\circ$.
1272. $16\sin 10^\circ\sin 30^\circ\sin 50^\circ\sin 70^\circ\sin 90^\circ$.
1273. $\tan 9^\circ-\tan 27^\circ-\tan 63^\circ+\tan 81^\circ$.

1274. Compute $\sin\alpha$, $\cos\alpha$, $\tan\alpha$ if $\cot\alpha=-2$ and $\frac{\pi}{2}<\alpha<\pi$.

1275. Compute $\sin\alpha$, $\tan\alpha$, $\cot\alpha$ if $\cos\alpha=-\frac{3}{5}$ and $\pi<\alpha<\frac{3\pi}{2}$.

1276. Compute $\cos\alpha$, $\tan\alpha$, $\cot\alpha$ if $\sin\alpha=-\frac{12}{13}$ and $\frac{3\pi}{2}<\alpha<2\pi$.

1277. Compute $\sin 2\alpha$, $\cos 2\alpha$, $\tan 2\alpha$, $\cot 2\alpha$ if $\cos\alpha=\frac{5}{13}$ and $0<\alpha<\frac{\pi}{2}$.

1278. Compute $\dfrac{5\sin\alpha+7\cos\alpha}{6\cos\alpha-3\sin\alpha}$ if $\tan\alpha=\frac{4}{15}$.

1279. Compute $\cos\left(\frac{\pi}{3}-\alpha\right)$ if $\sin\alpha=-\frac{12}{13}$ and $\frac{3}{2}\pi<\alpha<2\pi$.

1280. Prove that $\alpha+\beta=\frac{\pi}{4}$ if $\sin\alpha=\frac{1}{\sqrt{5}}$, $\sin\beta=\frac{1}{\sqrt{10}}$ and $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$.

1281. Find: (a) $\tan^2\alpha+\cot^2\alpha$; (b) $\tan^3\alpha+\cot^3\alpha$; (c) $\tan\alpha-\cot\alpha$ if $\tan\alpha+\cot\alpha=m$.

1282. Compute $\sin\frac{\alpha}{2}$, $\cos\frac{\alpha}{2}$, $\tan\frac{\alpha}{2}$ if (a) $\cos\alpha=0.8$ and $0<\alpha<\frac{\pi}{2}$; (b) $\tan\alpha=3\frac{3}{7}$ and $180^\circ<\alpha<270^\circ$.

1283. Compute $\sin\frac{\alpha}{4}$ if $\sin\alpha=\frac{336}{625}$ and $450^\circ<\alpha<540^\circ$.

1284. Prove that $\sin x=\frac{2a}{1+a^2}$, $\cos x=\frac{1-a^2}{1+a^2}$, $\tan x=\frac{2a}{1-a^2}$, $\cot x=\frac{1-a^2}{2a}$ if $\tan\frac{x}{2}=a$.

1285. Compute $\tan\frac{\alpha}{2}$ if $\sin\alpha+\cos\alpha=\frac{\sqrt{7}}{2}$ and $0<\alpha<\frac{\pi}{6}$.

In Problems 1286 through 1308, prove the indicated identities:

1286. $1+\cot\frac{\beta}{2}+\cos\left(45^\circ-\frac{\beta}{2}\right)=\cot\frac{\beta}{2}\cot\left(45^\circ-\frac{\beta}{2}\right)$.

1287. $\tan 3\alpha=\tan\alpha\tan(60^\circ+\alpha)\tan(60^\circ-\alpha)$.

1288. $\cos\alpha\cos 2\alpha\cos 4\alpha\cos 8\alpha\cos 16\alpha=\frac{\sin 32\alpha}{32\sin\alpha}$.

1289. $9\cos 15\alpha+3\cos 7\alpha+3\cos 19\alpha+9\cos 11\alpha=24\cos^3 2\alpha\cos 13\alpha$.

1290. $\frac{1+\sin\alpha\cos\alpha}{\sin^{-1}\alpha-\cos^{-1}\alpha-\sin\alpha+\cos\alpha}=\frac{\sqrt{2}\sin 2\alpha}{4\sin\left(\frac{\pi}{4}-\alpha\right)}$.

1291. $3-4\cos 2\alpha+\cos 4\alpha=8\sin^4\alpha$.

1292. $\sqrt{\frac{1}{1+\cos\alpha}+\frac{1}{1-\cos\alpha}}\sin\alpha=\sqrt{2}$ if $0<\alpha<\pi$.

1293. $\sqrt{\frac{1}{\sin^2\alpha}+\frac{1}{\cos^2\alpha}}=-\frac{2}{\sin 2\alpha}$ if $-\frac{\pi}{2}<\alpha<0$

1294. $\sqrt{\sin^2\alpha(1+\cot\alpha)+\cos^2\alpha(1+\tan\alpha)}=\sqrt{2}\cos\left(\alpha-\frac{\pi}{4}\right)$ if $-\frac{\pi}{4}\leqslant\alpha\leqslant\frac{3\pi}{4}$, $\alpha\neq 0$.

1295. $\sqrt{\cot\alpha+\cos\alpha}+\sqrt{\cot\alpha-\cos\alpha}=2\cos\frac{\alpha}{2}\sqrt{\cot\alpha}$ if $0<\alpha\leqslant\frac{\pi}{2}$.

1296. $$\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}-\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}=\left[\begin{array}{l}2\tan\alpha \text{ if } -\frac{\pi}{2}+2\pi k<\alpha<\frac{\pi}{2}+2\pi k\\ -2\tan\alpha \text{ if } \frac{\pi}{2}+2\pi k<\alpha<\frac{3\pi}{2}+2\pi k.\end{array}\right.$$

1297. $$\sqrt{\tan^2\alpha+\cos^2\alpha+2}=\left[\begin{array}{l}\frac{2}{\sin 2\alpha} \text{ if } \pi k<\alpha<\frac{\pi}{2}+\pi k\\ -\frac{2}{\sin 2\alpha} \text{ if } -\frac{\pi}{2}+\pi k<\alpha<\pi k.\end{array}\right.$$

1298. $$\sqrt{1+\cos 2\alpha}+\sqrt{1-\cos 2\alpha}+\sqrt{2}(\sin\alpha+\cos\alpha)$$
$$=\left[\begin{array}{ll}2\sqrt{2}(\sin\alpha+\cos\alpha) & \text{if } 2\pi k\leqslant\alpha\leqslant\frac{\pi}{2}+2\pi k\\ 2\sqrt{2}\sin\alpha & \text{if } \frac{\pi}{2}+2\pi k<\alpha<\pi+2\pi k\\ 0 & \text{if } \pi+2\pi k\leqslant\alpha\leqslant\frac{3}{2}\pi+2\pi k\\ 2\sqrt{2}\cos\alpha & \text{if } \frac{3\pi}{2}+2\pi k<\alpha<2\pi+2\pi k.\end{array}\right.$$

1299. $$\sqrt{1+2\sin\alpha\cos\alpha}=\left[\begin{array}{l}\sqrt{2}\cos\left(\alpha-\frac{\pi}{4}\right) \text{ if } -\frac{\pi}{4}+2\pi k\leqslant\alpha\leqslant\frac{3\pi}{4}+2\pi k\\ -\sqrt{2}\cos\left(\alpha-\frac{\pi}{4}\right) \text{ if } \frac{3\pi}{4}+2\pi k<\alpha<\frac{7\pi}{4}+2\pi k,\end{array}\right.$$

1300. $\tan 2\alpha \tan (30^\circ - \alpha) + \tan 2\alpha \tan (60^\circ - \alpha) + \tan (60^\circ - \alpha) \times \tan (30^\circ - \alpha) = 1$.

1301. $\sin^3 \alpha \sin^3 (\beta - \gamma) + \sin^3 \beta \sin^3 (\gamma - \alpha) + \sin^3 \gamma \sin^3 (\alpha - \beta) = 3 \sin \alpha \sin \beta \sin \gamma \sin (\alpha - \beta) \sin (\beta - \gamma) \sin (\gamma - \alpha)$.

1302. $\cos \alpha + \cos \beta + \cos \gamma = 1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$ if $\alpha + \beta + \gamma = \pi$.

1303. $\sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$ if $\alpha + \beta + \gamma = \pi$.

1304. $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$ if $\alpha + \beta + \gamma = \pi$.

1305. $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma - 1 = (-1)^n 2 \cos \alpha \cos \beta \cos \gamma$ if $\alpha + \beta + \gamma = \pi n$.

1306. $\sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2}$ if $\alpha + \beta = \gamma$.

1307. $\sin \alpha + \sin \beta + \sin \gamma + \sin \delta = 4 \sin \frac{\alpha+\beta}{2} \sin \frac{\beta+\gamma}{2} \sin \frac{\gamma+\alpha}{2}$

if $\alpha + \beta + \gamma + \delta = 2\pi$.

1308. $\cos^2 \alpha + \cos^2 \beta - \cos^2 \gamma - \cos^2 \delta = 2 \sin (\beta + \gamma) \sin (\alpha + \gamma) \sin (\alpha + \beta)$ if $\alpha + \beta + \gamma + \delta = 2\pi$.

In Problems 1309 through 1322, prove that

1309. If $\tan 2\alpha - \cot 2\beta - \cot 2\gamma = \tan 2\alpha \cot 2\beta \cot 2\gamma$, then $\alpha + \beta + \gamma = \frac{\pi}{2} n$.

1310. If $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$, then $\alpha + \beta + \gamma = \pi n$.

1311. If $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma - 2 = 2 \cos \alpha \cos \beta \cos \gamma$, then

$$\left[\begin{array}{l} \alpha+\beta+\gamma=\pi(2k+1) \\ \alpha-\beta+\gamma=\pi(2l+1) \\ \alpha+\beta-\gamma=\pi(2m+1) \\ \alpha-\beta-\gamma=\pi(2n+1). \end{array}\right.$$

1312. If $m \sin (\alpha + \beta) = \cos (\alpha - \beta)$, then $\frac{1}{1 - m \sin 2\alpha} + \frac{1}{1 - m \sin 2\beta} = \frac{2}{1 - m^2}$.

1313. If $\cos^2 \alpha + \cos^2 \beta = m$, then $\cos (\alpha + \beta) \cos (\alpha - \beta) = m - 1$.

1314. If $3 \sin \beta = \sin (2\alpha + \beta)$, then $\tan (\alpha + \beta) = 2 \tan \alpha$.

1315. If $\sin^2 \beta = \sin \alpha \cos \alpha$, then $\cos 2\beta = 2 \cos^2 \left(\frac{\pi}{4} + \alpha \right)$.

1316. If $\sin (2\alpha + \beta) = 2 \sin \beta$ and $\beta \neq \pi k$, then $\tan (\alpha + \beta) = 3 \tan \alpha$.

1317. If $\begin{cases} \sin \alpha - \cos \alpha = m \\ \sin 2\alpha = n - m^2 \end{cases}$, where $-\sqrt{2} \leqslant m \leqslant \sqrt{2}$, then $n = 1$.

1318. If $\begin{cases} \cos \alpha + \cos \beta = m \\ \sin \alpha + \sin \beta = n \end{cases}$, where $\begin{cases} m \neq 0 \\ n \neq 0 \end{cases}$, then $\sin (\alpha + \beta) = \frac{2mn}{m^2 + n^2}$.

1319. If $\begin{cases} m \cot \alpha = a \\ b \sin 2\alpha = n \end{cases}$, then $n (a^2 + m^2) = 2abm$.

1320. If $\begin{cases} \sin \alpha + \cos \alpha = m \\ \sin^3 \alpha + \cos^3 \alpha = n \end{cases}$, then $m^3 - 3m + 2n = 0$.

1321. If $\begin{cases} \cot \alpha + \tan \alpha = m \\ \frac{1}{\cos \alpha} - \cos \alpha = n \end{cases}$, then $m \sqrt[3]{m^2 n} - n \sqrt[3]{mn^2} = 1$.

1322. If $\begin{cases} a\cos^3\alpha+3a\cos\alpha\sin^2\alpha=m \\ a\sin^3\alpha+3a\cos^2\alpha\sin\alpha=n \end{cases}$, then $\sqrt[3]{(m+n)^2}+\sqrt[3]{(m-n)^2}=2\sqrt[3]{a}$.

SEC. 22. TRANSFORMING FUNCTIONS CONTAINING INVERSE TRIGONOMETRIC FUNCTIONS

Let us recall the definitions of inverse trigonometric functions.

(1) $y=\arcsin x$; this is a function defined on the interval $[-1, 1]$, and the inverse of the function $x=\sin y$, $y\in\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus,

$$(y=\arcsin x) \Leftrightarrow \begin{cases} -\frac{\pi}{2}\leqslant y\leqslant\frac{\pi}{2} \\ \sin y=x. \end{cases}$$

For any x from the interval $[-1, 1]$ we have:

$$-\frac{\pi}{2}\leqslant\arcsin x\leqslant\frac{\pi}{2}, \tag{1}$$

$$\sin(\arcsin x)=x. \tag{2}$$

(2) $y=\arccos x$; this is a function defined on the interval $[-1, 1]$, and the inverse of the function $x=\cos y$, $y\in[0, \pi]$, Thus,

$$(y=\arccos x) \Leftrightarrow \begin{cases} 0\leqslant y\leqslant\pi \\ \cos y=x. \end{cases}$$

For any x from the interval $[-1, 1]$ we have:

$$0\leqslant\arccos x\leqslant\pi, \tag{3}$$

$$\cos(\arccos x)=x. \tag{4}$$

(3) $y=\arctan x$; this is a function defined on the interval $(-\infty, \infty)$, and the inverse of the function $x=\tan y$, $y\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Thus,

$$(y=\arctan x) \Leftrightarrow \begin{cases} -\frac{\pi}{2}<y<\frac{\pi}{2} \\ \tan y=x. \end{cases}$$

For any x we have:

$$-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}, \tag{5}$$

$$\tan(\arctan x) = x. \tag{6}$$

(4) $y = \operatorname{arccot} x$; this is a function defined on the interval $(-\infty, \infty)$, and the inverse of the function $x = \cot y$, $y \in (0, \pi)$. Thus

$$(y = \operatorname{arccot} x) \Leftrightarrow \begin{cases} 0 < y < \pi \\ \cot y = x. \end{cases}$$

For any x we have:

$$0 < \operatorname{arccot} x < \pi, \tag{7}$$

$$\cot(\operatorname{arccot} x) = x. \tag{8}$$

The functions $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$, $y = \operatorname{arccot} x$ are called *inverse trigonometric functions* or *arc functions.*

Note the following fundamental identities:

$$\begin{aligned} \arcsin(-x) &= -\arcsin x, \ (-1 \leqslant x \leqslant 1), \\ \arccos(-x) &= \pi - \arccos x \ (-1 \leqslant x \leqslant 1), \\ \arctan(-x) &= -\arctan x, \\ \operatorname{arccot}(-x) &= \pi - \operatorname{arccot} x. \end{aligned}$$

Consider several examples.

Example 1. Simplify the function $\cos(\arcsin x)$, where $-1 \leqslant x \leqslant 1$.

Solution. Let us set $\arcsin x = y$. Then $\sin y = x$, $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$. Now, to find $\cos y$, we take advantage of the relationship $\cos^2 y = 1 - \sin^2 y$. Hence, $\cos^2 y = 1 - x^2$. But $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$, and on this interval cosine takes on only nonnegative values.

Thus, $\cos y = \sqrt{1 - x^2}$, that is, $\cos(\arcsin x) = \sqrt{1 - x^2}$, where $-1 \leqslant x \leqslant 1$.

Example 2. Simplify the function $\cos(2\arcsin x)$.

Solution. $\cos(2\arcsin x) = \cos^2(\arcsin x) - \sin^2(\arcsin x) = (1 - x^2) - x^2 = 1 - 2x^2$.

Example 3. Simplify the function $\sin(\arctan x)$.

Solution. We set $y = \arctan x$, then $\tan y = x$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$. To find $\cos y$, let us use the equality $\cos^2 y = \frac{1}{1 + \tan^2 y}$. But $-\frac{\pi}{2} < y < \frac{\pi}{2}$, and on this interval cosine takes on only positive

values. Therefore $\cos y = \frac{1}{\sqrt{1+\tan^2 y}}$, that is, $\cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$.

Since $\sin y = \tan y \cdot \cos y$, we have: $\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$.

Example 4. Compute $\sin\left(\frac{1}{2}\operatorname{arccot}\left(-\frac{3}{4}\right)\right)$.

Solution. Let $\operatorname{arccot}\left(-\frac{3}{4}\right) = \alpha$. Then $\cot\alpha = -\frac{3}{4}$, $0 < \alpha < \pi$ $\left(\text{more precisely, } \frac{\pi}{2} < \alpha < \pi, \text{ since } \cot\alpha < 0\right)$. We have to compute $\sin\frac{\alpha}{2}$. We have: $\tan\alpha = -\frac{4}{3}$.

Using the formula $1+\tan^2\alpha = \frac{1}{\cos^2\alpha}$, we find: $\cos^2\alpha = \frac{9}{25}$. But, by hypothesis, $\frac{\pi}{2} < \alpha < \pi$, and in this interval $\cos\alpha < 0$, consequently, $\cos\alpha = -\frac{3}{5}$.

Knowing $\cos\alpha$, we can find $\sin\frac{\alpha}{2}$, using the formula $1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$. We get: $\sin^2\frac{\alpha}{4} = \frac{4}{5}$, whence $\sin\frac{\alpha}{2} = \frac{2}{\sqrt{5}}$ or $\sin\frac{\alpha}{2} = -\frac{2}{\sqrt{5}}$. But $\frac{\pi}{4} < \frac{\alpha}{2} < \frac{\pi}{2}$, and in this interval sine takes on only positive values. Thus,

$$\sin\left(\frac{1}{2}\operatorname{arccot}\left(-\frac{3}{4}\right)\right) = \frac{2}{\sqrt{5}}.$$

Example 5. Compute $\arccos\left(\cos\left(-\frac{17}{5}\pi\right)\right)$.

Let us set $y = \arccos\left(\cos\left(-\frac{17}{5}\pi\right)\right)$. Then $\cos y = \cos\left(-\frac{17}{5}\pi\right)$, $0 \leqslant y \leqslant \pi$. We have: $\cos\left(-\frac{17}{5}\pi\right) = \cos\left(-4\pi + \frac{3}{5}\pi\right) = \cos\frac{3}{5}\pi$. Thus, $\cos\frac{3}{5}\pi = \cos y$, and since $0 < \frac{3}{5}\pi < \pi$, $y = \frac{3}{5}\pi$.

Remark. The equality $\arccos\left(\cos\left(-\frac{17}{5}\pi\right)\right) = -\frac{17}{5}\pi$ would be untrue since arccosine does not attain the value equal to $\left(-\frac{17}{5}\pi\right)$ (see (3)).

Example 6. Prove that

$$\arccos\frac{1}{2} + \arccos\left(-\frac{1}{7}\right) = \arccos\left(-\frac{13}{14}\right). \tag{9}$$

Proof. Let us set $\alpha=\arccos\frac{1}{2}$, $\beta=\arccos\left(-\frac{1}{7}\right)$, $\gamma=\arccos\left(-\frac{13}{14}\right)$. Then $\alpha=\frac{\pi}{3}$; $\cos\beta=-\frac{1}{7}$, $\frac{\pi}{2}<\beta<\pi$; $\cos\gamma=-\frac{13}{14}$, $\frac{\pi}{2}<\gamma<\pi$.

Let us prove that $\alpha+\beta=\gamma$. To this end, we consider the equality $T(\alpha+\beta)=T(\gamma)$, where T is a trigonometric function. But the equality $T(\alpha+\beta)=T(\gamma)$, generally speaking, does not yet imply the equality $\alpha+\beta=\gamma$ (for instance, $\sin 30°=\sin 150°$, but $30°\neq 150°$). The equality $\alpha+\beta=\gamma$ will take place if $\alpha+\beta$ and γ belong to the same monotonicity interval of the function T.

In the example under consideration γ belongs to the second quadrant, while $\alpha+\beta$ either to the second or to the third quadrant, that is, γ and $\alpha+\beta$ belong to the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Therefore, it is expedient to take as T such a trigonometric function which is monotone on the indicated interval. Such a function is, for instance, sine. Thus, let us find $\sin(\alpha+\beta)$. We have:

$$\begin{aligned}\sin(\alpha+\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\&=\sin\frac{\pi}{3}\times\left(-\frac{1}{7}\right)+\cos\frac{\pi}{3}\times\sqrt{1-\cos^2\beta}\\&=-\frac{\sqrt{3}}{14}+\frac{4\sqrt{3}}{14}=\frac{3\sqrt{3}}{14}.\end{aligned}$$

Thus, $\sin(\alpha+\beta)=\frac{3\sqrt{3}}{14}$. Let us now compute $\sin\gamma$. We have:

$$\sin\gamma=\sqrt{1-\cos^2\gamma}=\sqrt{1-\left(-\frac{13}{14}\right)^2}=\frac{3\sqrt{3}}{14}.$$

Thus, we get:

$$\sin(\alpha+\beta)=\sin\gamma. \tag{10}$$

Since $\alpha+\beta$ and γ belong to the same monotonicity interval of sine, it follows from Equality (10) that $\alpha+\beta=\gamma$. Thereby Equality (9) has been proved.

Example 7. Let us prove that if $-1<x<1$, then

$$\arcsin x=\arctan\frac{x}{\sqrt{1-x^2}}. \tag{11}$$

Proof. Let us compute the values of tangent of both sides of Equality (11). We get:

$$\tan(\arcsin x)=\frac{x}{\sqrt{1-x^2}},\quad \tan\left(\arctan\frac{x}{\sqrt{1-x^2}}\right)=\frac{x}{\sqrt{1-x^2}},$$

that is, the tangents are equal. Further, $-\frac{\pi}{2}<\arcsin x<\frac{\pi}{2}$ (the inequalities are strict since, by hypothesis, $-1<x<1$) and $-\frac{\pi}{2}<\arctan\frac{x}{\sqrt{1-x^2}}<\frac{\pi}{2}$, that is, $\arcsin x$ and $\arctan x$ belong to the same monotonicity interval of tangent. Thereby Identity (11) has been proved.

EXERCISES

In Problems 1323 through 1339, compute the given expressions:

1323. $2\arcsin\left(-\frac{\sqrt{3}}{2}\right)+\operatorname{arccot}(-1)+\arccos\frac{1}{\sqrt{2}}+\frac{1}{2}\arccos(-1)$.

1324. $\tan\left(5\arctan\frac{\sqrt{3}}{3}-\frac{1}{4}\arcsin\frac{\sqrt{3}}{2}\right)$.

1325. $\sin\left(3\arctan\sqrt{3}+2\arccos\frac{1}{2}\right)$.

1326. $\cos\left(3\arcsin\frac{\sqrt{3}}{2}+\arccos\left(-\frac{1}{2}\right)\right)$.

1327. $\arccos\left(\cos\frac{\pi}{4}\right)$. **1328.** $\arctan(\tan 0.3\pi)$.

1329. $\arcsin\left(-\sin\frac{7}{3}\pi\right)$. **1330.** $\arccos\left(-\cos\frac{3\pi}{4}\right)$.

1331. $\arctan\left(-\tan\frac{2\pi}{3}\right)$. **1332.** $\arcsin\left(\sin\frac{33\pi}{7}\right)+\arccos\left(\cos\frac{46\pi}{7}\right)$.

1333. $\arctan\left(-\tan\frac{13\pi}{8}\right)+\operatorname{arccot}\left(\cot\left(-\frac{19\pi}{8}\right)\right)$.

1334. $\sin\left(\frac{1}{2}\arcsin\left(-\frac{2\sqrt{2}}{3}\right)\right)$. **1335.** $\tan\left(\frac{1}{2}\arcsin\frac{5}{13}\right)$.

1336. $\cot\left(\frac{1}{2}\arccos\left(-\frac{4}{7}\right)\right)$. **1337.** $\sin\left(\arctan\frac{8}{15}-\arcsin\frac{8}{17}\right)$.

1338. $\cos\left(2\arctan\frac{1}{4}+\arccos\frac{3}{5}\right)$.

1339. $\sin\left(2\left(\arcsin\frac{\sqrt{5}}{3}-\arccos\frac{\sqrt{5}}{3}\right)\right)$.

In Problems 1340 through 1350, simplify the indicated functions:

1340. $\cos(\arccos x+\arccos y)$. **1341.** $\sin(\arccos x+\arcsin y)$.
1342. $\tan(\arctan x+\arctan y)$. **1343.** $\tan(\arcsin x+\arcsin y)$.
1344. $\sin(2\arcsin x)$. **1345.** $\tan(2\arctan x)$.
1346. $\cos(2\arctan x)$. **1347.** $\sin(2\operatorname{arccot} x)$. **1348.** $\cos(2\operatorname{arccot} x)$.
1349. $\cos\left(\frac{1}{2}\arccos x\right)$. **1350.** $\tan\left(\frac{1}{2}\arctan x\right)$.

In Problems 1351 through 1358, check the given equalities:

1351. $\arctan \frac{2}{3} + \arctan \frac{1}{5} = \frac{\pi}{4}$. **1352.** $\operatorname{arccot} \frac{1}{9} + \operatorname{arccot} \frac{4}{5} = \frac{3\pi}{4}$.

1353. $\operatorname{arccot} \frac{1}{7} + 2 \operatorname{arccot} \frac{1}{3} = \frac{5\pi}{4}$.

1354. $\arcsin \frac{4}{5} - \arccos \frac{2}{\sqrt{5}} = \arctan \frac{1}{2}$.

1355. $\arcsin \frac{7}{25} + \frac{1}{2} \arccos \frac{7}{25} = \arccos \frac{3}{5}$.

1356. $\arctan \frac{\sqrt{2}}{2} + \arcsin \frac{\sqrt{2}}{2} = \arctan (3+2\sqrt{2})$.

1357. $\arctan \frac{1}{3} + \arctan \frac{1}{4} + \arctan \frac{2}{9} = \frac{\pi}{4}$.

1358. $\arcsin \frac{4}{5} + \arcsin \frac{5}{13} + \arcsin \frac{16}{65} = \frac{\pi}{2}$.

In Problems 1359 through 1369, prove the given identities:

1359. $\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}}$.

1360. $\arcsin x = \begin{cases} \arccos \sqrt{1-x^2} & \text{if } 0 \leqslant x \leqslant 1, \\ -\arccos \sqrt{1-x^2} & \text{if } -1 \leqslant x \leqslant 0. \end{cases}$

1361. $\arccos x = \begin{cases} \arcsin \sqrt{1-x^2} & \text{if } 0 \leqslant x \leqslant 1, \\ \pi - \arcsin \sqrt{1-x^2} & \text{if } -1 \leqslant x \leqslant 0. \end{cases}$

1362. $\arctan x = \begin{cases} \arccos \frac{1}{\sqrt{1+x^2}} & \text{if } x > 0, \\ -\arccos \frac{1}{\sqrt{1+x^2}} & \text{if } x \leqslant 0. \end{cases}$

1363. $\arccos x = \begin{cases} \arctan \frac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \leqslant 1, \\ \pi + \arctan \frac{\sqrt{1-x^2}}{x} & \text{if } -1 \leqslant x < 0. \end{cases}$

1364. $\arctan x = \begin{cases} \operatorname{arccot} \frac{1}{x} & \text{if } x > 0, \\ \operatorname{arccot} \frac{1}{x} - \pi & \text{if } x < 0. \end{cases}$

1365. $\arcsin x = \begin{cases} \operatorname{arccot} \frac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \leqslant 1, \\ \operatorname{arccot} \frac{\sqrt{1-x^2}}{x} - \pi & \text{if } -1 \leqslant x < 0. \end{cases}$

1366. $\operatorname{arccot} x = \begin{cases} \arcsin \dfrac{1}{\sqrt{1+x^2}} & \text{if } x \geqslant 0, \\ \pi - \arcsin \dfrac{1}{\sqrt{1+x^2}} & \text{if } x < 0. \end{cases}$

1367. $\operatorname{arccot} x = \begin{cases} \arctan \dfrac{1}{x} & \text{if } x > 0, \\ \pi + \arctan \dfrac{1}{x} & \text{if } x < 0. \end{cases}$

1368. $2 \arccos \sqrt{\dfrac{1+x}{2}} = \arccos x.$

1369. $\dfrac{1}{2} \arccos (2x^2 - 1) = \arccos x$ if $x \geqslant 0$.

SEC. 23. PROVING INEQUALITIES

When proving trigonometric inequalities, we usually use the same methods as for proving algebraic inequalities (see Sec. 5). Here we should like to note that when proving trigonometric inequalities by the synthetic method, we frequently use the following inequalities as reference ones:

$$|\sin x| \leqslant 1, \quad |\cos x| \leqslant 1, \quad \sin x < x < \tan x, \quad \text{where } 0 < x < \frac{\pi}{2}.$$

Sometimes we use as reference inequalities those following from the monotonicity of trigonometric functions. Thus, in the interval $\left(0, \frac{\pi}{2}\right)$ the functions $y = \sin x$ and $y = \tan x$ increase, while the functions $y = \cos x$ and $y = \cot x$ decrease. Therefore, if $0 < x_1 < x_2 < \frac{\pi}{2}$, then $\sin x_1 < \sin x_2$, $\cos x_1 > \cos x_2$, $\tan x_1 < \tan x_2$, $\cot x_1 > \cot x_2$. Similar inequalities can be obtained for other intervals of monotonicity of trigonometric functions.

Consider several examples.

Example 1. Prove the inequality $a \sin^2 \alpha + \frac{b}{\sin^2 \alpha} \geqslant 2\sqrt{ab}$ if it is known that $a > 0$, $b > 0$, $\alpha \neq \pi n$.

Proof. Let us use the inequality relating the arithmetic mean and the geometric mean of two positive numbers a_1 and a_2: $\frac{a_1 + a_2}{2} \geqslant \sqrt{a_1 a_2}$.

Let us set in this inequality $a \sin^2 \alpha = a_1$, $\frac{b}{\sin^2 \alpha} = a_2$. We get:

$$\frac{a \sin^2 \alpha + \dfrac{b}{\sin^2 \alpha}}{2} \geqslant \sqrt{a \sin^2 \alpha \cdot \frac{b}{\sin^2 \alpha}},$$

whence $a \sin^2 \alpha + \frac{b}{\sin^2 \alpha} \geqslant 2\sqrt{ab}$, which was required to be proved.

Example 2. Prove that if A, B, C are angles of a triangle, then

$$\cos A + \cos B + \cos C \leqslant \frac{3}{2}. \qquad (1)$$

Proof. Let us carry out some transformations of the left-hand side of Inequality (1). We have:

$$\cos A + \cos B + \cos C = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + \cos C.$$

Since, by hypothesis, $A + B + C = 180°$, we have: $\frac{A+B}{2} = 90° - \frac{C}{2}$, and, consequently,

$$\cos \frac{A+B}{2} = \cos \left(90° - \frac{C}{2}\right) = \sin \frac{C}{2}.$$

Since $0 \leqslant \cos \frac{A-B}{2} \leqslant 1$, we have: $\cos \frac{A+B}{2} \cos \frac{A-B}{2} \leqslant \sin \frac{C}{2}$. Thus,

$$2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + \cos C \leqslant 2 \sin \frac{C}{2} + \cos C.$$

Consider the expression $2 \sin \frac{C}{2} + \cos C$. We have:

$$2 \sin \frac{C}{2} + \cos C = 2 \sin \frac{C}{2} + 1 - 2 \sin^2 \frac{C}{2}.$$

Let us set $x = \sin \frac{C}{2}$. Then

$$2 \sin \frac{C}{2} + 1 - 2 \sin^2 \frac{C}{2} = -2x^2 + 2x + 1.$$

If we now prove that $-2x^2 + 2x + 1 \leqslant \frac{3}{2}$, then the validity of Inequality (1) follows.

Consider the parabola given by the equation $y = -2x^2 + 2x + 1$. This parabola opens downward, the point $\left(\frac{1}{2}, \frac{3}{2}\right)$ being its vertex. Hence, the inequality $-2x^2 + 2x + 1 \leqslant \frac{3}{2}$ is true for any x.

Thus, Inequality (1) has been proved.

Example 3. Prove the inequality

$$\sin \alpha \sin 2\alpha \sin 3\alpha < \frac{3}{4}. \qquad (2)$$

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Proof. Let us carry out some transformations of the left-hand side of Inequality (2). We have:

$$(\sin\alpha \sin 2\alpha)\sin 3\alpha = \frac{\cos\alpha - \cos 3\alpha}{2}\sin 3\alpha$$

$$= \frac{2\sin 3\alpha\cos\alpha - 2\sin 3\alpha\cos 3\alpha}{4} = \frac{\sin 4\alpha + \sin 2\alpha - \sin 6\alpha}{4}.$$

Since $\sin 4\alpha \leqslant 1$, $\sin 2\alpha \leqslant 1$, $-\sin 6\alpha \leqslant 1$, we have: $\sin 4\alpha + \sin 2\alpha - \sin 6\alpha \leqslant 3$, the equality sign taking place only for those values of α which satisfy the system of equations

$$\begin{cases}\sin 4\alpha = 1\\ \sin 2\alpha = 1\\ \sin 6\alpha = -1.\end{cases}$$

But this system has no solution. Indeed, if $\sin 2\alpha = 1$, then $\cos 2\alpha = 0$, and therefore $\sin 4\alpha = 2\sin 2\alpha\cos 2\alpha = 0$.

Thus, $\sin 4\alpha + \sin 2\alpha - \sin 6\alpha < 3$, and, hence, $\frac{\sin 4\alpha + \sin 2\alpha - \sin 6\alpha}{4} < \frac{3}{4}$, whence there just follows Inequality (2).

Example 4. Prove that

$$\tan\alpha_1 < \frac{\sin\alpha_1 + \sin\alpha_2 + \ldots + \sin\alpha_n}{\cos\alpha_1 + \cos\alpha_2 + \ldots + \cos\alpha_n} < \tan\alpha_n$$

if $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \frac{\pi}{2}$.

Proof. Since in the interval $\left(0, \frac{\pi}{2}\right)$ the function $y = \sin x$ increases, and the function $y = \cos x$ decreases, we have:

$$0 < \sin\alpha_1 < \sin\alpha_2 < \ldots < \sin\alpha_n,$$

$$\cos\alpha_1 > \cos\alpha_2 > \ldots > \cos\alpha_n > 0.$$

Hence, $n\sin\alpha_1 < \sin\alpha_1 + \sin\alpha_2 + \ldots + \sin\alpha_n < n\sin\alpha_n$, $n\cos\alpha_1 > \cos\alpha_1 + \cos\alpha_2 + \ldots + \cos\alpha_n > n\cos\alpha_n$, whence $\frac{n\sin\alpha_1}{n\cos\alpha_1} < \frac{\sin\alpha_1 + \sin\alpha_2 + \ldots + \sin\alpha_n}{\cos\alpha_1 + \cos\alpha_2 + \ldots + \cos\alpha_n} < \frac{n\sin\alpha_n}{n\cos\alpha_n}$, i.e. $\tan\alpha_1 < \frac{\sin\alpha_1 + \sin\alpha_2 + \ldots + \sin\alpha_n}{\cos\alpha_1 + \cos\alpha_2 + \ldots + \cos\alpha_n} < \tan\alpha_n$, which was required to be proved.

Example 5. Prove that

$$\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\alpha\tan\gamma < 1$$

if $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\alpha + \beta + \gamma < \frac{\pi}{2}$.

Proof. Let us set $\frac{\pi}{2}-\alpha-\beta=\gamma_1$. Then $\gamma<\gamma_1$, since, by hypothesis, $\gamma<\frac{\pi}{2}-\alpha-\beta$. Consider the expression $\tan\alpha\tan\beta+\tan\beta\tan\gamma_1+\tan\alpha\tan\gamma_1$. It is possible to prove (see Example 11, Sec. 21) that

$$\tan\alpha\tan\beta+\tan\beta\tan\gamma_1+\tan\alpha\tan\gamma_1=1.$$

But γ and γ_1 are arguments from the first quadrant $(\gamma<\gamma_1)$. Hence, $\tan\gamma<\tan\gamma_1$ and therefore

$$\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\alpha\tan\gamma<\tan\alpha\tan\beta+\tan\beta\tan\gamma_1 +\tan\alpha\tan\gamma_1,$$

that is, $\tan\alpha\tan\beta+\tan\beta\tan\alpha+\tan\alpha\tan\gamma<1$, which was required to be proved.

Example 6. Prove that if α, β, γ are angles of a triangle, then $(\sin\alpha+\sin\beta+\sin\gamma)^2>9\sin\alpha\sin\beta\sin\gamma$.

Proof. Let us use the inequality relating the arithmetic mean and the geometric mean of three positive numbers: $\frac{a+b+c}{3}\geqslant\sqrt{abc}$.

Setting in this inequality $\sin\alpha=a$, $\sin\beta=b$, $\sin\gamma=c$, we get:

$$\frac{\sin\alpha+\sin\beta+\sin\gamma}{3}\geqslant\sqrt[3]{\sin\alpha\sin\beta\sin\gamma},$$

and, further, $(\sin\alpha+\sin\beta+\sin\gamma)^2\geqslant 9\sqrt[3]{(\sin\alpha\sin\beta\sin\gamma)^2}$, but, $\sqrt[3]{(\sin\alpha\sin\beta\sin\gamma)^2}>\sqrt[3]{(\sin\alpha\sin\beta\sin\gamma)^3}=\sin\alpha\sin\beta\sin\gamma$.

Thus, $(\sin\alpha+\sin\beta+\sin\gamma)^2>9\sin\alpha\sin\beta\sin\gamma$, which was required to be proved.

Example 7. Prove the inequality

$$\alpha-\frac{\alpha^3}{4}<\sin\alpha, \tag{3}$$

where $0<\alpha<\frac{\pi}{2}$.

Proof. Let us take $\frac{\alpha}{2}<\tan\frac{\alpha}{2}$ as a reference inequality. Transforming it, we get:

$$\alpha\cos\frac{\alpha}{2}<2\sin\frac{\alpha}{2}, \quad \alpha\cos\frac{\alpha}{2}\cos\frac{\alpha}{2}<2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2},$$

$$\alpha\cos^2\frac{\alpha}{2}<\sin\alpha, \quad \alpha\left(1-\sin^2\frac{\alpha}{2}\right)<\sin\alpha.$$

Let us now use the inequality

$$\sin\frac{\alpha}{2}<\frac{\alpha}{2}. \tag{4}$$

Since, by hypothesis, $0<\alpha<\frac{\pi}{2}$, we have: $\sin\frac{\alpha}{2}>0$ and $\frac{\alpha}{2}>0$, therefore Inequality (4) can be transformed to

$$\sin^2\frac{\alpha}{2}<\frac{\alpha^2}{4}, \text{ i.e. } 1-\sin^2\frac{\alpha}{2}>1-\frac{\alpha^2}{4},$$

whence

$$\alpha\left(1-\sin^2\frac{\alpha}{2}\right)>\alpha-\frac{\alpha^3}{4}. \qquad (5)$$

Comparing Inequalities (3) and (5), we get:

$$\alpha-\frac{\alpha^3}{4}<\alpha\left(1-\sin^2\frac{\alpha}{2}\right)<\sin\alpha,$$

whence $\alpha-\frac{\alpha^3}{4}<\sin\alpha$, which was required to be proved.

Example 8. Prove the inequality

$$\tan\alpha-\alpha<\tan\beta-\beta \qquad (6)$$

if $0<\alpha<\beta<\frac{\pi}{2}$.

Proof. Let us use the inequality $\tan x>x$, where $0<x<\frac{\pi}{2}$. Let us set $x=\beta-\alpha$. Then $\tan(\beta-\alpha)>\beta-\alpha$. If we prove that

$$\tan\beta-\tan\alpha>\tan(\beta-\alpha), \qquad (7)$$

thereby we shall prove the inequality $\tan\beta-\tan\alpha>\beta-\alpha$, and, consequently, Inequality (6).

Thus, let us prove Inequality (7). For this purpose, we form the difference $(\tan\beta-\tan\alpha)-\tan(\beta-\alpha)$ and then transform it:

$$\tan\beta-\tan\alpha-\tan(\beta-\alpha)=\tan\beta-\tan\alpha-\frac{\tan\beta-\tan\alpha}{1+\tan\alpha\tan\beta}$$
$$=\frac{\tan\alpha\tan\beta}{1+\tan\alpha\tan\beta}(\tan\beta-\tan\alpha).$$

The obtained expression is positive since $\tan\beta>\tan\alpha$.

Hence, it follows that Inequality (7) and, hence, Inequality (6) are true.

Example 9. Prove the inequality

$$\cos\alpha+3\cos3\alpha+6\cos6\alpha\geqslant-7\frac{3}{16}. \qquad (8)$$

Proof. Suppose the contrary, that is,

$$\cos\alpha+3\cos3\alpha+6\cos6\alpha<-7\frac{3}{16}. \qquad (9)$$

Transforming Inequality (9), we get:

$$\cos\alpha+3\cos 3\alpha+6\cos 6\alpha+6<-1\frac{3}{16},$$

$$\cos\alpha+3\cos 3\alpha+12\cos^2 3\alpha<-1\frac{3}{16},$$

$$\cos\alpha+3(\cos 3\alpha+4\cos^2 3\alpha)<-1\frac{3}{16},$$

$$3\left(4\cos^2 3\alpha+\cos 3\alpha+\frac{1}{16}\right)+\cos\alpha-\frac{3}{16}<-1\frac{3}{16},$$

$$3\left(2\cos 3\alpha+\frac{1}{4}\right)^2+\cos\alpha<-1. \qquad (10)$$

Inequality (10) is false since $\left(2\cos 3\alpha+\frac{1}{4}\right)^2\geqslant 0$, and $\cos\alpha\geqslant -1$. Hence, our supposition is not true, and, therefore, Inequality (8) is true.

Example 10. Prove the inequality

$$\cos 36^\circ>\tan 36^\circ. \qquad (11)$$

Proof. Suppose that Inequality (11) is not true, that is, that $\cos 36^\circ\leqslant\tan 36^\circ$. Then, we get:

$$\cos 36^\circ\leqslant\frac{\sin 36^\circ}{\cos 36^\circ},$$

$$\cos^2 36^\circ\leqslant\sin 36^\circ,$$

$$1+\cos 72^\circ\leqslant 2\sin 36^\circ,$$

$$1+\cos(90^\circ-18^\circ)\leqslant 2\sin(6^\circ+30^\circ),$$

$$1+\sin 18^\circ\leqslant 2\sin 6^\circ\cos 30^\circ+2\sin 30^\circ\cos 6^\circ,$$

$$1+2\sin 9^\circ\cos 9^\circ\leqslant\cos 6^\circ+2\sin 6^\circ\cos 30^\circ. \qquad (12)$$

Since $1>\cos 6^\circ$, $\sin 9^\circ>\sin 6^\circ$, $\cos 9^\circ>\cos 30^\circ$, Inequality (12) is not true. Hence, Inequality (11) is true.

Example 11. Prove that if A, B, C are angles of a triangle, then

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leqslant\frac{1}{8}. \qquad (13)$$

Proof. Suppose that Inequality (13) is not true, i.e. that $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}>\frac{1}{8}$.

Then, when transforming the product $\sin\frac{A}{2}\sin\frac{B}{2}$ to a half-difference of cosines, we get:

$$\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)\sin\frac{C}{2}>\frac{1}{4},$$

and, further,

$$\cos\frac{A-B}{2}\sin\frac{C}{2}-\cos\frac{A+B}{2}\sin\frac{C}{2}>\frac{1}{4},$$

$$\sin\frac{A-B+C}{2}+\sin\frac{-A+B+C}{2}-\sin\frac{A+B+C}{2}$$
$$+\sin\frac{A+B-C}{2}>\frac{1}{2}. \qquad (14)$$

Since $A+B+C=180°$, we have: $A-B+C=180°-2B$, $-A+B+C=180°-2A$, $A+B-C=180°-2C$, and, therefore,

$$\sin\frac{A+C-B}{2}=\cos B, \quad \sin\frac{C-A+B}{2}=\cos A,$$

$$\sin\frac{A+B+C}{2}=1, \quad \sin\frac{A+B-C}{2}=\cos C.$$

This enables us to rewrite Inequality (14) as follows:

$$\cos B+\cos A-1+\cos C>\frac{1}{2},$$

or

$$\cos A+\cos B+\cos C\geqslant\frac{3}{2}.$$

And this contradicts the inequality proved in Example 2, Sec. 23. Hence, our supposition is not true, and, therefore, Inequality (13) is true.

Example 12. Prove the inequality

$$\tan n\alpha > n\tan\alpha \qquad (15)$$

if $0<\alpha<\frac{\pi}{4(n-1)}$, where n is a natural number, $n\neq 1$.

Proof. Let us apply the method of mathematical induction.

(1) Check the validity of Inequality (15) for $n=2$, that is, prove that

$$\tan 2\alpha > 2\tan\alpha, \qquad (16)$$

where $0<\alpha<\frac{\pi}{4}$.

Indeed, $\tan 2\alpha-2\tan\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}-2\tan\alpha=2\tan\alpha\frac{\tan^2\alpha}{1-\tan^2\alpha}$.

For $0<\alpha<\frac{\pi}{4}$ we have: $\tan\alpha>0$, $1-\tan^2\alpha>0$, and, hence,

$$2\tan\alpha\frac{\tan^2\alpha}{1-\tan^2\alpha}>0.$$

Hence it follows that Inequality (16) is true.

(2) Suppose that Inequality (15) is true for $n=k$, that is, $\tan k\alpha > k\tan\alpha$, where $0<\alpha<\frac{\pi}{4(k-1)}$. Let us prove that Inequality (15) is also true for $n=k+1$, that is,

$$\tan(k+1)\alpha > (k+1)\tan\alpha, \tag{17}$$

where $0<\alpha<\frac{\pi}{4k}$. Indeed,

$$\tan(k+1)\alpha=\frac{\tan k\alpha+\tan\alpha}{1-\tan k\alpha\tan\alpha}>\frac{k\tan\alpha+\tan\alpha}{1-\tan k\alpha\tan\alpha}.$$

By hypothesis, $0<\alpha<\frac{\pi}{4k}$, hence, $\tan k\alpha<\tan\frac{\pi}{4}=1$, and $\tan\alpha<1$. But then $\frac{k\tan\alpha+\tan\alpha}{1-\tan k\alpha\tan\alpha}>(k+1)\tan\alpha$, whence it follows that Inequality (17) is valid.

By the principle of mathematical induction, we conclude that Inequality (15) is true for any natural $n\geqslant 2$.

EXERCISES

In Problems 1370 through 1431, prove the given inequalities:

1370. $\sqrt{\cos\varphi}<\sqrt{2}\cos\frac{\varphi}{2}$ if $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$.

1371. $\cot\frac{\alpha}{2}\geqslant 1+\cot\alpha$ if $0<\alpha<\pi$.

1372. $\tan\alpha\tan\beta<1$ if α, β are sizes of the acute angles of an obtuse triangle.

1373. $\tan\alpha\tan\beta>1$ if α, β are sizes of the angles of an acute triangle.

1374. $\cos\alpha+\cos\beta>\cos\gamma$ if $\alpha>0$, $\beta>0$, $\gamma>0$, $\alpha+\beta+\gamma=\frac{\pi}{2}$.

1375. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma\leqslant 2$ if α, β, γ are sizes of the angles of a non-acute triangle.

1376. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma>2$ if α, β, γ are sizes of the angles of an acute triangle.

1377. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma\geqslant 1$ if α, β, γ are sizes of the angles of a non-acute triangle.

1378. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma<1$ if α, β, γ are sizes of the angles of an acute triangle.

1379. $\sin\frac{\alpha+\beta}{2}\geqslant\frac{\sin\alpha+\sin\beta}{2}$ if $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$.

1380. $\cos\frac{\alpha+\beta}{2}\geqslant\frac{\cos\alpha+\cos\beta}{2}$ if $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$.

1381. $\tan\frac{\alpha+\beta}{2}\leqslant\frac{\tan\alpha+\tan\beta}{2}$ if $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$.

1382. $\sin\left(\frac{\pi}{2}-\cos\alpha\right)>0$. **1383.** $\sin^4\alpha+\cos^4\alpha\geqslant\frac{1}{2}$.

1384. $\sin^6\alpha+\cos^6\alpha\geqslant\frac{1}{4}$. **1385.** $\sin^8\alpha+\cos^8\alpha\geqslant\frac{1}{8}$.

1386. $\sin^{2n}\alpha+\cos^{2n}\alpha\leqslant 1$.

1387. $\tan(\alpha+\beta)>\tan\alpha+\tan\beta$ if $\alpha>0$, $\beta>0$, $\alpha+\beta<\frac{\pi}{2}$

1388. $\sin^4 x-6\sin^2 x+5>0$.

1389. $\cos(\sin x)>0$. **1390.** $\sin(2+\cos x)>0$.

1391. $\cos(\pi+\arcsin x)\leqslant 0$. **1392.** $\sin\left(\frac{\pi}{2}+\arctan x\right)>0$.

1393. $\tan\alpha+\cot\alpha\geqslant 2$ if $0<\alpha<\frac{\pi}{2}$.

1394. $-\sqrt{2}\leqslant\sin\alpha+\cos\alpha\leqslant\sqrt{2}$.

1395. $\tan\alpha+\cot\alpha>\sin\alpha+\cos\alpha$ if $0<\alpha<\frac{\pi}{2}$.

1396. $|\tan\alpha+\cot\alpha|>|\sin\alpha+\cos\alpha|$.

1397. $\sin(\alpha+\beta)\leqslant\sin\alpha+\sin\beta$ if $0<\alpha<\pi$, $0<\beta<\pi$.

1398. $\cos(\alpha-\beta)\leqslant\cos\alpha+\sin\beta$ if $0<\alpha<\pi$, $0<\beta<\pi$.

1399. $\sin(\alpha+\beta+\gamma)<\sin\alpha+\sin\beta+\sin\gamma$ if $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$, $0<\gamma<\frac{\pi}{2}$.

1400. $\frac{\sin\alpha+\tan\alpha}{\cos\alpha+\cot\alpha}>0$ if $\alpha\neq\frac{\pi k}{2}$.

1401. $\frac{3}{(1+\sin^2\alpha)(1+\cos^2\alpha)}<2+\tan^2\alpha+\cot^2\alpha$ if $\alpha\neq\frac{\pi k}{2}$.

1402. $3(\tan^2\alpha+\cot^2\alpha)-8(\tan\alpha+\cot\alpha)+10\geqslant 0$.

1403. $\frac{\sin\alpha-1}{\sin\alpha-2}+\frac{1}{2}\geqslant\frac{2-\sin\alpha}{3-\sin\alpha}$.

1404. $(1+\sin^2\alpha)(\tan^2\alpha-2)+\cot^2\alpha+\cos^2\alpha\geqslant 0$.

1405. $\sin 2\alpha\cos 2\alpha\cos 4\alpha\cos 8\alpha\cos 16\alpha\leqslant\frac{1}{16}$

1406. $\cos\alpha\cos 2\alpha\cos 4\alpha\cdot\ldots\cdot\cos 2^n\alpha\leqslant\frac{1}{2^{n+1}\sin\alpha}$ if $0<\alpha<\pi$.

1407. $-\frac{1}{4}\leqslant\sin\alpha\sin\left(\frac{\pi}{3}-\alpha\right)\sin\left(\frac{\pi}{3}+\alpha\right)\leqslant\frac{1}{4}$

1408. $0\leqslant\cos^2\alpha+\cos^2(\alpha+\beta)-2\cos\alpha\cos\beta\cos(\alpha+\beta)\leqslant 1$.

1409. $\tan\alpha(\cot\beta+\cot\gamma)+\tan\beta(\cot\alpha+\cot\gamma)+\tan\gamma(\cot\alpha+\cot\beta)\geqslant 6$ if $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$, $0<\gamma<\frac{\pi}{2}$.

1410. $(\cot^2\alpha-1)(3\cot^2\alpha-1)(\cot 3\alpha\tan 2\alpha-1)\leqslant -1$.

1411. $(1-\tan^2\alpha)(1-3\tan^2\alpha)(1+\tan 2\alpha\tan 3\alpha)>0$.

1412. $\sin^2\alpha+\sin^2\beta\geqslant\sin\alpha\sin\beta+\sin\alpha+\sin\beta-1$.

1413. $\alpha-\sin\alpha<\beta-\sin\beta$ if $0<\alpha<\beta<\frac{\pi}{2}$.

1414. $\frac{\sin\alpha}{\alpha} > \frac{\sin\beta}{\beta}$ if $0 < \alpha < \beta < \frac{\pi}{2}$.

1415. $\cos\alpha + 2\sin\alpha > 1$ if $0 < \alpha \leqslant \frac{\pi}{2}$.

1416. $\frac{\tan\alpha}{\alpha} > \frac{\alpha}{\sin\alpha}$ if $0 < \alpha < \frac{\pi}{2}$.

1417. $\frac{\sin\alpha + \tan\alpha}{2} > \alpha$ if $0 < \alpha < \frac{\pi}{2}$.

1418. $\tan\frac{\alpha}{2} < \alpha$ if $0 < \alpha \leqslant \frac{\pi}{2}$.

1419. $\sin\alpha + \sin 2\alpha + \ldots + \sin n\alpha < n$.

1420. $\frac{\sin\alpha + \sin 3\alpha + \ldots + \sin(2n-1)\alpha}{\sin\alpha} \geqslant 0$.

1421. $(1 - \sin\alpha)^2 + \sin^2(\alpha - 1) > 0$. **1422.** $\frac{\sin\alpha}{\sin\alpha - \cos\alpha\tan\frac{\alpha}{2}} < 2$.

1423. $\cos(\alpha+\beta)\cos(\alpha-\beta) \leqslant \cos^2\alpha$. **1424.** $\frac{1}{\sin^4\alpha} + \frac{1}{\cos^4\alpha} \geqslant 8$.

1425. $\tan^2\alpha + \tan^2\beta + \tan^2\gamma \geqslant 1$ if $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha + \beta + \gamma = \frac{\pi}{2}$.

1426. $\sin^2\alpha + \sin^2\beta + \sin^2\gamma \geqslant \frac{3}{4}$ if $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\alpha + \beta + \gamma = \frac{\pi}{2}$.

1427. $4\sin 3\alpha + 5 \geqslant 4\cos 2\alpha + 5\sin\alpha$. **1428.** $|\sin n\alpha| \leqslant n\,|\sin\alpha|$.

1429. $\cos\alpha \leqslant \cos^{2n}\frac{\alpha}{2^n}$ if $0 < \alpha < \frac{\pi}{2}$.

1430. $\left(1 + \frac{1}{\sin^n\alpha}\right)\left(1 + \frac{1}{\cos^n\alpha}\right) \geqslant \left(1 + 2^{\frac{n}{2}}\right)^2$ if $0 < \alpha < \frac{\pi}{2}$.

1431. $\cos 2\gamma \leqslant 0$ if $\tan\gamma = \frac{1}{\cos\alpha\cos\beta} + \tan\alpha\tan\beta$.

Chapter 4

SOLVING EQUATIONS AND INEQUALITIES

SEC. 24. EQUATIONS

Let us recall the general formulas for solving simplest trigonometric equations (it is supposed here that the parameters n, k, l, m, ... are integers unless otherwise stated).

Equation	Solution
$\sin x = a$, where $\lvert a \rvert \leqslant 1$	$x = (-1)^k \arcsin a + \pi k$
$\cos x = a$, where $\lvert a \rvert \leqslant 1$	$x = \pm \arccos a + 2\pi k$
$\tan x = a$	$x = \arctan a + \pi k$
$\cot x = a$	$x = \operatorname{arccot} a + \pi k$

Let us point out some particular cases of simplest trigonometric equations whose solutions can be written without using the general formulas:

$$\sin x = 0 \Leftrightarrow x = \pi k,$$

$$\sin x = 1 \Leftrightarrow x = \frac{\pi}{2} + 2\pi k,$$

$$\sin x = -1 \Leftrightarrow x = -\frac{\pi}{2} + 2\pi k,$$

$$\cos x = 0 \Leftrightarrow x = \frac{\pi}{2} + \pi k,$$

$$\cos x = 1 \Leftrightarrow x = 2\pi k,$$

$$\cos x = -1 \Leftrightarrow x = \pi + 2\pi k,$$

$$\tan x = 0 \Leftrightarrow x = \pi k.$$

Found solutions must be necessarily checked if in the process of solving an equation the following took place:

(1) an extension of the domain of definition of the equation* as a result of some transformations (getting rid of denominators, reducing a fraction, collecting like terms),

(2) both of its sides were raised to the same even power,

(3) trigonometric identities were used whose left-hand and right-hand sides had different domains of definition, for instance,

$$\frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\sin\alpha,\quad \frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\cos\alpha,$$

$$\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}=\tan\alpha,\quad \tan\alpha\cot\alpha=1,$$

$$\frac{1-\cos\alpha}{\sin\alpha}=\tan\frac{\alpha}{2},$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\ \text{ and others.}$$

When used from left to right, these identities lead to an extension of the domain of definition of an equation, and, hence, may cause the appearance of extraneous roots. When used from right to left, these identities lead to a contraction of the domain of definition of an equation, which is, generally speaking, inadmissible, due to a possible loss of roots.

For example, consider the equation

$$\tan\left(x+\frac{\pi}{4}\right)=2\cot x-1. \tag{1}$$

We have:

$$\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+1}{1-\tan x}, \tag{2}$$

$$\cot x=\frac{1}{\tan x}. \tag{3}$$

Then Equation (1) is transformed to

$$\frac{\tan x+1}{1-\tan x}=\frac{2}{\tan x}-1.$$

Setting $y=\tan x$, we get: $\frac{y+1}{1-y}=\frac{2}{y}-1$, whence we find: $y=\frac{1}{2}$, that is, $\tan x=\frac{1}{2}$, and, consequently, $x=\arctan\frac{1}{2}+\pi n$.

* The intersection of domains of definition of two functions $f(x)$ and $g(x)$ will be called the domain of definition of the equation $f(x)=g(x)$.

This family satisfies Equation (1). But it is easy to note that the values $x = \frac{\pi}{2} + \pi k$ also satisfy Equation (1). The reason of the loss of solutions is the application of Identities (2) and (3). The replacement of the function $\tan\left(x + \frac{\pi}{4}\right)$ by the function $\frac{\tan x + 1}{1 - \tan x}$ as well as the replacement of $\cot x$ by $\frac{1}{\tan x}$ reduce the domain of definition of Equation (1), namely, the values $x = \frac{\pi}{2} + \pi k$ "slip out" from the domain of definition. And these values are the "lost" solutions of Equation (1).

When solving trigonometric equations, we use two basic methods: factorization and introduction of new variables.

When solving equations by introducing new variables, we should remember an important role played by the choice of a function in terms of which the rest of functions are expressed. One choice of such a function can lead to an irrational equation, and the other to a rational equation. It is clear that the second choice is preferable. Setting, for instance, $y = \sin x$ in the equation $2\cos^2 x + 4\cos x = 3\sin^2 x$, we get the collection of two irrational equations:

$$2(1 - y^2) + 4\sqrt{1 - y^2} = 3y^2;$$
$$2(1 - y^2) - 4\sqrt{1 - y^2} = 3y^2.$$

If we set $y = \cos x$, then we get a rational equation: $2y^2 + 4y = 3(1 - y^2)$.

We shall denote by $R(\cos x, \sin x)$ a rational function of $\cos x$ and $\sin x$, that is, a function including addition, multiplication, and division of $\cos x$, $\sin x$, and constants.

Consider the equation of the form: $R(\cos x, \sin x) = 0$. In some cases we succeed in reducing such an equation to a rational equation with respect to $\sin x$ (or $\cos x$). Let us give some rules promoting the choice of the substitution for solving trigonometric equations. If $\cos x$ enters an equation only in even powers, then, replacing everywhere $\cos^2 x$ by $1 - \sin^2 x$, we get a rational equation with respect to $\sin x$. In similar fashion, if $\sin x$ enters an equation only in even powers, then the replacement of $\sin^2 x$ by $1 - \cos^2 x$ results in the rational form of the equation with respect to $\cos x$.

A *homogeneous trigonometric equation of the first degree* is defined as an equation of the form:

$$a \sin x + b \cos x = 0.$$

A *homogeneous trigonometric equation of the second degree* is defined as an equation of the form:

$$a \sin^2 x + b \sin x \cos x + c \cos^2 x = 0.$$

Likewise, we may define a homogeneous trigonometric equation of any natural power n.

Consider the case when $a \neq 0$. It is easy to see that for $a \neq 0$ the homogeneous equation is not satisfied by those values of x for which $\cos x = 0$. Therefore the division by $\cos x$ (by $\cos^2 x$) of both sides of a homogeneous equation of the first (second) degree when $a \neq 0$ leads to an equivalent equation. Let us divide both sides of the homogeneous equation of the first degree by $\cos x$, and both sides of the homogeneous equation of the second degree by $\cos^2 x$. As a result, we get the following equations rational with respect to $\tan x$, and, therefore, solved by the substitution $z = \tan x$:

$$a \tan x + b = 0, \quad a \tan^2 x + b \tan x + c = 0.$$

Let us now consider the substitution enabling us to reduce any equation of the form $R(\cos x, \sin x) = 0$ to a rational equation. This substitution is $u = \tan \frac{x}{2}$.

If $x \neq \pi + 2\pi k$, then the following identities hold true:

$$\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \quad \sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}.$$

Therefore the substitution $u = \tan \frac{x}{2}$ transforms the equation $R(\cos x, \sin x) = 0$ into the equation

$$R\left(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}\right) = 0.$$

The left-hand side of the last equation is a rational function. Hence, our substitution has led the equation to the rational form. The substitution $u = \tan \frac{x}{2}$ is said to be *universal*. Since the use of the universal substitution is possible only for $x \neq \pi + 2\pi k$, we have to check whether or not the numbers of the form $x = \pi + 2\pi k$ are solutions of the given equation.

In this section, in addition to trigonometric equations in one variable we are going to consider equations in two and three variables, as well as equations containing an inverse trigonometric function of the variable. We shall do this by way of examples.

Example 1. Solve the equation $\sin 5x + \sin x + 2 \sin^2 x = 1$.

Solution. Since $\sin 5x + \sin x = 2 \sin 3x \cos 2x$, $2 \sin^2 x = 1 - \cos 2x$, the equation takes the form:

$$2 \sin 3x \cos 2x + 1 - \cos 2x = 1,$$

and, further, $\cos 2x\,(2 \sin 3x - 1) = 0$.

The problem has been reduced to the collection of equations $\cos 2x = 0$; $2 \sin 3x - 1 = 0$.

From these equations we find two families of solutions:

$$x = \frac{\pi}{4} + \frac{\pi}{2} k; \quad x = (-1)^n \frac{\pi}{18} + \frac{\pi n}{3}.$$

Example 2. Solve the equation $\cos 15x = \sin 5x$.

Solution. Since $\cos 15x = \sin\left(\frac{\pi}{2} + 15x\right)$, we rewrite the equation as follows:

$$\left(\sin \frac{\pi}{2} + 15x\right) - \sin 5x = 0.$$

Hence, $2 \sin\left(5x + \frac{\pi}{4}\right) \cos\left(10x + \frac{\pi}{4}\right) = 0$.

Consequently, $\sin\left(5x + \frac{\pi}{4}\right) = 0$; $\cos\left(10x + \frac{\pi}{4}\right) = 0$.

From the first equation of the collection we get: $5x + \frac{\pi}{4} = \pi k$, whence $x = -\frac{\pi}{20} + \frac{\pi}{5} k$, from the second equation of the collection we get: $10x + \frac{\pi}{4} = \frac{\pi}{2} + \pi n$, whence $x = \frac{\pi}{40} + \frac{\pi}{10} n$.

Example 3. Solve the equation $\cos 4x \cos 8x - \cos 5x \cos 9x = 0$.

Solution. Let us transform the products of cosines into sums

$$\frac{\cos 12x + \cos 4x}{2} - \frac{\cos 14x + \cos 4x}{2} = 0,$$

and, further,

$$\frac{1}{2}(\cos 12x - \cos 14x) = 0,$$

whence $\sin 13x \cdot \sin x = 0$.

Now, the problem is reduced to solving the collection of equations $\sin 13x = 0$; $\sin x = 0$ from which we find two families of solutions of the given equation: $x = \frac{\pi}{13} k$; $x = \pi n$.

The set $\left\{\frac{\pi}{13} k\right\}$ contains the set $\{\pi n\}$ (it suffices to set $k = 13n$). Therefore the answer may be written more briefly: $x = \frac{\pi}{13} k$.

Example 4. Solve the equation

$$\sin x + 7 \cos x = 5. \tag{4}$$

Solution. First Method. Dividing both sides of Equation (4) by $\sqrt{1^2 + 7^2} = \sqrt{50}$, we get:

$$\frac{1}{\sqrt{50}} \sin x + \frac{7}{\sqrt{50}} \cos x = \frac{5}{\sqrt{50}}. \tag{5}$$

Since $\left(\frac{1}{\sqrt{50}}\right)^2+\left(\frac{7}{\sqrt{50}}\right)^2=1$, there is a value of φ such that $\frac{1}{\sqrt{50}}=\sin\varphi$, $\frac{7}{\sqrt{50}}=\cos\varphi$, where $\varphi=\arcsin\frac{1}{\sqrt{50}}$ is an auxiliary angle $\left(\text{or } \varphi=\arccos\frac{7}{\sqrt{50}}\right)$. Now, Equation (5) can be rewritten as follows:

$$\sin\varphi\sin x+\cos x\cos\varphi=\frac{\sqrt{2}}{2} \quad \text{or} \quad \cos(x-\varphi)=\frac{\sqrt{2}}{2},$$

whence $x-\varphi=\pm\frac{\pi}{4}+2\pi k$.

Since $\varphi=\arcsin\frac{1}{\sqrt{50}}$, we finally obtain the following solutions of Equation (4):

$$x=\pm\frac{\pi}{4}+\arcsin\frac{1}{\sqrt{50}}+2\pi k \left(\text{or } x=\pm\frac{\pi}{4}+\arccos\frac{7}{\sqrt{50}}+2\pi k\right).$$

Second Method. Let us solve Equation (4) with the aid of the universal substitution. Expressing $\sin x$ and $\cos x$ in terms of $\tan\frac{x}{2}$ and setting $u=\tan\frac{x}{2}$, we come to the rational equation

$$\frac{2u}{1+u^2}+\frac{7(1-u^2)}{1+u^2}=5.$$

Solving this equation, we find: $u_1=\frac{1}{2}$, $u_2=-\frac{1}{3}$.

Now, we have to solve the following collection of equations:

$$\tan\frac{x}{2}=\frac{1}{2}; \quad \tan\frac{x}{2}=-\frac{1}{3}.$$

From these equations we find:

$$x=2\arctan\frac{1}{2}+2\pi k; \quad x=2\arctan\left(-\frac{1}{3}\right)+2\pi n.$$

A check shows that the values $x=\pi+2\pi m$ do not satisfy Equation (4) (the necessity to check these values when the universal substitution is used has been already mentioned).

Thus, Equation (4) has the following solutions:

$$x=2\arctan\frac{1}{2}+2\pi k; \; x=-2\arctan\frac{1}{3}+2\pi n.$$

Example 5. Solve the equation

$$5\sin x-12\cos x=-13\sin 3x. \tag{6}$$

Solution. As in Example 4, we apply the method of introducing an auxiliary angle. Dividing both sides of Equation (6) by $\sqrt{5^2+12^2}=13$, we get:

$$\frac{5}{13}\sin x-\frac{12}{13}\cos x=-\sin 3x. \tag{7}$$

Since $\left(\frac{5}{13}\right)^2+\left(\frac{12}{13}\right)^2=1$, there is a value of φ such that $\frac{5}{13}=\cos\varphi$ and $\frac{12}{13}=\sin\varphi$ $\left(\text{or } \frac{5}{13}=\sin\varphi \text{ and } \frac{12}{13}=\cos\varphi\right)$.

Now, Equation (7) may be rewritten in the following way:

$$\sin x\cos\varphi-\cos x\sin\varphi=-\sin 3x,$$

and, further,

$$\sin(x-\varphi)+\sin 3x=0,$$

$$2\sin\left(2x-\frac{\varphi}{2}\right)\cos\left(x+\frac{\varphi}{2}\right)=0.$$

Solving the collection of equations

$$\sin\left(2x-\frac{\varphi}{2}\right)=0;\quad \cos\left(x+\frac{\varphi}{2}\right)=0,$$

we get: $x=\frac{\varphi}{4}+\frac{\pi}{2}k$; $x=-\frac{\varphi}{2}+\frac{\pi}{2}+\pi n$.

Taking into consideration that $\varphi=\arcsin\frac{12}{13}$, we get the following two families of solutions of Equation (6):

$$x=\frac{1}{4}\arcsin\frac{12}{13}+\frac{\pi}{2}k;\ x=-\frac{1}{2}\arcsin\frac{12}{13}+\frac{\pi}{2}+\pi n.$$

Example 6. Solve the equation

$$\frac{\sin 2x}{\sin\frac{2x+\pi}{3}}=0. \tag{8}$$

Solution. Since the fraction is equal to zero, it follows from the equation that $\sin 2x=0$, whence $x=\frac{\pi}{2}k$. Of the found solutions, the original equation is satisfied by those and only by those solutions which belong to the domain of definition of the original equation. The domain of definition of Equation (8) is given by the condition $\sin\frac{2x+\pi}{3}\neq 0$, whence

$$x\neq\frac{3\pi n-\pi}{2}. \tag{9}$$

Let us mark the found solutions $\left(x = \frac{\pi}{2} k\right)$ with points on the number line (Fig. 37) and cross out the points excluded by Condi-

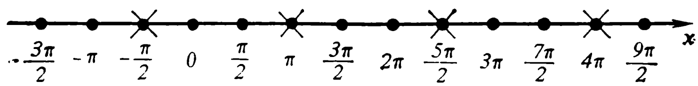

Fig. 37

tion (9). As a result, we get the following solutions of Equation (8):

$$x = \frac{3\pi}{2} l; \quad x = \frac{\pi}{2} (3m + 1).$$

These two families of solutions may be written more briefly: $x = \frac{\pi}{2} l$, where $l \neq 3m - 1$ and $m \in \boldsymbol{Z}$.

Example 7. Solve the equation

$$8 \sin x - 7 \cos x = 0. \tag{10}$$

Solution. Equation (10) is a homogeneous equation of the first degree. Dividing both sides of the equation by $\cos x$ (this transformation will not lead to a loss of roots), we get: $8 \tan x - 7 = 0$, whence $x = \arctan \frac{7}{8} + \pi k$.

Example 8. Solve the equation

$$\sin^2 x + 2 \sin x \cos x - 3 \cos^2 x = 0.$$

Solution. Dividing both sides of this homogeneous equation of the second degree by $\cos^2 x$ (this transformation will not lead to a loss of roots), we get: $\tan^2 x + 2 \tan x - 3 = 0$.

Setting $u = \tan x$, we arrive at the quadratic equation $u^2 + 2u - 3 = 0$, whence we find: $u_1 = -3$, $u_2 = 1$.

Solving the collection of equations $\tan x = -3$; $\tan x = 1$, we get:

$$x = \arctan(-3) + \pi k; \quad x = \frac{\pi}{4} + \pi n.$$

Example 9. Solve the equation

$$5 \sin^2 x + 3 \sin x \cos x - 3 \cos^2 x = 2.$$

Solution. This equation is not homogeneous since the right-hand side of the equation is different from zero. It can be transformed into a homogeneous equation. To this end, we use the identity $\sin^2 x + \cos^2 x = 1$. Then the equation can be rewritten in the following way:

$$5 \sin^2 x + 3 \sin x \cos x - 3 \cos^2 x = 2 (\sin^2 x + \cos^2 x),$$

and, further,

$$3 \sin^2 x + 3 \sin x \cos x - 5 \cos^2 x = 0.$$

The last equation represents a homogeneous equation of the second degree. Dividing it by $\cos^2 x$ and using the substitution $u = \tan x$, we get:

$$x = \arctan \frac{-3 \pm \sqrt{69}}{6} + \pi k.$$

Example 10. Solve the equation

$$5 \sin^2 x + \sqrt{3} \sin x \cos x + 6 \cos^2 x = 5.$$

Solution. We have:

$$5 \sin^2 x + \sqrt{3} \sin x \cos x + 6 \cos^2 x = 5 (\sin^2 x + \cos^2 x),$$

$$\sqrt{3} \sin x \cos x + \cos^2 x = 0. \qquad (11)$$

The obtained equation contains no term $a \sin^2 x$, that is, $a = 0$. Here, we are not allowed to divide both sides of the equation by $\cos^2 x$, since those values of x for which $\cos^2 x = 0$ satisfy Equation (11), and therefore the division by $\cos^2 x$ will lead to a loss of roots. We shall proceed in a different way: we shall factorize the left-hand member of Equation (11) into factors. We get: $\cos x (\sqrt{3} \sin x + \cos x) = 0$.

Now, the problem is reduced to solving the collection of equations:

$$\cos x = 0; \quad \sqrt{3} \sin x + \cos x = 0. \qquad (12)$$

From the first equation of Collection (12) we find: $x = \frac{\pi}{2} + \pi k$.

Dividing both sides of the second equation of Collection (12) by $\cos x$, we get: $\tan x = -\frac{\sqrt{3}}{3}$, whence $x = -\frac{\pi}{6} + \pi n$.

Thus, we have obtained two families of solutions of Equation (11):

$$x = \frac{\pi}{2} + \pi k; \quad x = -\frac{\pi}{6} + \pi n.$$

Example 11. Solve the equation

$$2 \cos^2 x + 4 \cos x = 3 \sin^2 x. \qquad (13)$$

Solution. Equation (13) contains only an even power of $\sin x$, therefore it is advisable to replace $\sin^2 x$ by $1 - \cos^2 x$ and then to set $u = \cos x$. Then, Equation (13) will take the form:

$$5u^2 + 4u - 3 = 0,$$

whence $u_{1,2} = \frac{-2 \pm \sqrt{19}}{5}$.

It remains to solve the collection of equations:

$$\cos x = \frac{-2-\sqrt{19}}{5}; \quad \cos x = \frac{-2+\sqrt{19}}{5}.$$

The first equation of this collection has no solution since $\left|\frac{-2-\sqrt{19}}{5}\right| > 1$, and from the second equation of the collection we find $x = \pm \arccos \frac{-2+\sqrt{19}}{5} + 2\pi k$ which is the solution of Equation (13).

Example 12. Solve the equation

$$\sin 2x + 5 \sin x + 5 \cos x + 1 = 0.$$

Solution. Setting $u = \sin x + \cos x$, we get:

$$u^2 = (\sin x + \cos x)^2 \text{ or } u^2 = 1 + \sin 2x.$$

Therefore the given equation will take the form: $u^2 + 5u = 0$, whence $u_1 = 0$, $u_2 = -5$.

Now, the problem has been reduced to solving the collection of equations: $\sin x + \cos x = 0$; $\sin x + \cos x = -5$. With both of its sides divided by $\cos x$, the first equation of the collection (a homogeneous equation of the first degree) is transformed to $\tan x + 1 = 0$, whence $x = -\frac{\pi}{4} + \pi k$. The second equation of the collection has no solution since $|\sin x| \leqslant 1$, $|\cos x| \leqslant 1$, and therefore the sum $\sin x + \cos x$ cannot be equal to the number -5.

Thus, the original equation has the following solution: $x = -\frac{\pi}{4} + \pi k$.

Example 13. Solve the equation

$$3 \tan 2x - 4 \tan 3x = \tan^2 3x \tan 2x. \tag{14}$$

Solution. Let us transform Equation (14) to

$$3 \tan 2x - 3 \tan 3x = \tan 3x + \tan^2 3x \tan 2x,$$

and, further,

$$3 (\tan 2x - \tan 3x) = \tan 3x (1 + \tan 3x \tan 2x). \tag{15}$$

Dividing both sides of Equation (15) by $1 + \tan 3x \tan 2x$, we get:

$$3 \frac{\tan 2x - \tan 3x}{1 + \tan 2x \tan 3x} = \tan 3x \tag{16}$$

or

$$-3 \tan x = \tan 3x, \tag{16'}$$

and, further, $\frac{3\sin x}{\cos x}+\frac{\sin 3x}{\cos 3x}=0$,

$$\sin 3x\cos x+3\sin x\cos 3x=0,$$
$$(\sin 3x\cos x+\sin x\cos 3x)+2\sin x\cos 3x=0,$$
$$\sin 4x+(\sin 4x-\sin 2x)=0,$$
$$4\sin 2x\cos 2x-\sin 2x=0,$$
$$\sin 2x\,(4\cos 2x-1)=0,$$
$$\sin 2x=0;\quad \cos 2x=\frac{1}{4},$$
$$x=\frac{\pi}{2}k;\quad x=\pm\frac{1}{2}\arccos\frac{1}{4}+\pi n.$$

Check. It is clear that Equations (14) and (15) are equivalent. Let us find out whether or not the passage from Equation (15) to Equation (16) was an equivalent transformation. For this purpose, let us find those values of x for which the expression $1+\tan 3x\tan 2x$ vanishes. We have:

$$1+\tan 3x\tan 2x=0, \tag{17}$$
$$\frac{\sin 3x\sin 2x}{\cos 3x\cos 2x}+1=0,$$
$$\sin 3x\sin 2x+\cos 3x\cos 2x=0,$$
$$\cos x=0,\quad x=\frac{\pi}{2}+\pi l.$$

These values of x do not satisfy Equation (17) ($\tan 3x$ is not defined for these values of x). Hence, Equation (17) has no solution, and therefore the function $1+\tan 3x\tan 2x$ is different from zero for any admissible values of x. This means that the division of both sides of Equation (15) by $1+\tan 3x\tan 2x$ was an equivalent transformation.

The rest of the transformations used for solving Equation (14) could lead only to the appearance of extraneous solutions (because of an extension of the domain of definition of the equation when we got rid of denominators or as a result of applying Formula (VI.3) when we replaced (16) by (16′)). The extraneous solutions are rejected with the aid of the domain of definition of Equation (14) which is determined by the following conditions: $\begin{cases}\cos 2x\neq 0\\ \cos 3x\neq 0.\end{cases}$

We have to reject the solution obtained for odd k's from the family $x=\frac{\pi}{2}k$, whereas the second family satisfies the indicated conditions. Thus, the solution of Equation (14) has the form:

$$x=\pi m;\quad x=\pm\frac{1}{2}\arccos\frac{1}{4}+\pi n.$$

Example 14. Solve the equation

$$\sin x + 2\sin 2x = 3 + \sin 3x. \tag{18}$$

Solution. We transform Equation (18) to

$$(\sin x - \sin 3x) + 2\sin 2x = 3,$$

and, further, $2\sin x\cos 2x - 2\sin 2x + 3 = 0$.

Let us now carry out the following transformations to get perfect squares:

$$(\sin^2 x + 2\sin x\cos 2x + \cos^2 2x) + (\sin^2 2x - 2\sin 2x + 1) + 3$$
$$= \sin^2 x + \cos^2 2x + \sin^2 2x + 1,$$

that is,

$$(\sin x + \cos 2x)^2 + (\sin 2x - 1)^2 + 3 = \sin^2 x + 2,$$

whence

$$(\sin x + \cos 2x)^2 + (\sin 2x - 1)^2 + \cos^2 x = 0. \tag{19}$$

But the sum of squares is equal to zero if and only if each term is equal to zero. Therefore Equation (19) is equivalent to the following system of equations:

$$\begin{cases} \sin x + \cos 2x = 0 \\ \sin 2x - 1 = 0 \\ \cos x = 0. \end{cases} \tag{20}$$

Solving the third equation of System (20) (the simplest one), we get: $x = \frac{\pi}{2} + \pi k$. Substituting these values into the second equation of the system, we shall have:

$$\sin 2\left(\frac{\pi}{2} + \pi k\right) - 1 = \sin(\pi + 2\pi k) - 1 = -1 \neq 0,$$

that is, the values $x = \frac{\pi}{2} + \pi k$ do not satisfy the second equation of System (20). Therefore System (20) is incompatible; thus, Equation (18) has no solution.

Example 15. Solve the equation

$$\sqrt{-3 - \cos^2 x + 3\sin 5x} = 1 - \sin x. \tag{21}$$

Solution. Squaring both sides of Equation (21) and collecting like terms, we get:

$$2\sin x + 3\sin 5x = 5. \tag{22}$$

Since $\sin x \leqslant 1$, $\sin 5x \leqslant 1$, Equation (22) is satisfied by those and only those values of x for which we have simultaneously: $\sin x = 1$ and $\sin 5x = 1$. In other words, Equation (22) is equivalent to

the following system of equations:

$$\begin{cases} \sin x = 1 \\ \sin 5x = 1. \end{cases} \tag{23}$$

Let us solve this system. From the equation $\sin x = 1$ we find: $x = \frac{\pi}{2} + 2\pi k$.

Substituting these values of x into the left-hand side of the second equation of System (23) we get:

$$\sin 5\left(\frac{\pi}{2} + 2\pi k\right) = \sin\left(\frac{5\pi}{2} + 10\pi k\right) = \sin\frac{\pi}{2} = 1.$$

Thus, $x = \frac{\pi}{2} + 2\pi k$ is the solution of System (23), and, hence, also of Equation (22).

The squaring of both sides of Equation (21) might cause extraneous solutions, therefore a check is necessary. In this case it is readily carried out by substituting the found values into the original equation (21). We have:

$\sqrt{-3 - \cos^2\left(\frac{\pi}{2} + 2\pi k\right) + 3\sin 5\left(\frac{\pi}{2} + 2\pi k\right)} = 0$ on the left-hand side of the equation, $1 - \sin\left(\frac{\pi}{2} + 2\pi k\right) = 0$ on the right-hand side of the equation.

Hence, $x = \frac{\pi}{2} + 2\pi k$ is the solution of Equation (21).

Example 16. Solve the equation

$$\sqrt{1 + \sin 2x} = \sqrt{2}\cos 2x. \tag{24}$$

Solution. Squaring both sides of Equation ((24), we get: $1 + \sin 2x = 2\cos^2 2x$. Let us set $u = \sin 2x$, then $\cos^2 2x = 1 - u^2$, and we come to the equation $1 + u = 2(1 - u^2)$, whence we find: $u_1 = -1, u_2 = \frac{1}{2}$. The problem has been reduced to solving the collection of equations: $\sin 2x = -1$; $\sin 2x = \frac{1}{2}$. From the first equation of this collection we find: $x = -\frac{\pi}{4} + \pi k$, from the second: $x = (-1)^n \frac{\pi}{12} + \frac{\pi}{2} n$.

Since we used squaring, we might get extraneous roots. Hence, the found solutions should be checked. In the present case the check is readily carried out with the aid of the condition $\cos 2x \geqslant 0$ (on this condition the squaring yields an equivalent equation).

We first check the values $x=-\frac{\pi}{4}+\pi k$. We have: $\cos 2x=\cos\left(-\frac{\pi}{2}+2\pi k\right)=0$. This means that the numbers of the form $x=-\frac{\pi}{4}+\pi k$ satisfy the condition $\cos 2x \geqslant 0$ and therefore are solutions of Equation (24).

Let us now check the values $x=(-1)^n\frac{\pi}{12}+\frac{\pi}{2}n$. We have: $\cos 2x=\cos\left((-1)^n\frac{\pi}{6}+\pi n\right)$. We shall assign the parameter n the values 0, 1, 2, 3 and so forth:

for $n=0$ $\cos\frac{\pi}{6}>0$,

for $n=1$ $\cos\left(-\frac{\pi}{6}+\pi\right)<0$,

for $n=2$ $\cos\left(\frac{\pi}{6}+2\pi\right)>0$,

for $n=3$ $\cos\left(-\frac{\pi}{6}+3\pi\right)<0$ and so on.

We note that $\cos 2x>0$ for an even n and $\cos 2x<0$ for an odd n. Analogously, this conclusion is valid for $n=-1, -2, -3, \ldots$.

Thus, from the values of the form $x=(-1)^n\frac{\pi}{12}+\frac{\pi n}{2}$ we have to take only the values corresponding to even n's, that is, the numbers of the form $n=2k$. Then we get: $x=\frac{\pi}{12}+\pi k$. Thus, Equation (24) has the following solutions: $x=-\frac{\pi}{4}+\pi k$; $x=\frac{\pi}{12}+\pi k$.

Example 17. Solve the equation

$$\arccos x-\arcsin x=\frac{\pi}{6}. \tag{25}$$

Solution. Let us find the cosines of both sides of the equation:

$$\cos(\arccos x-\arcsin x)=\cos\frac{\pi}{6}. \tag{26}$$

We get:

$$x\sqrt{1-x^2}+\sqrt{1-x^2}\,x=\frac{\sqrt{3}}{2};$$

$$4x\sqrt{1-x^2}=\sqrt{3}, \tag{27}$$

$$16x^4-16x^2+3=0, \tag{28}$$

whence

$$x_1=\frac{1}{2},\quad x_2=-\frac{1}{2},\quad x_3=\frac{\sqrt{3}}{2},\quad x_4=-\frac{\sqrt{3}}{2}.$$

When solving Equation (25), we twice carried out transformations which might cause the appearance of extraneous solutions, namely, we took cosine when passing from Equation (25) to Equation (26) and squared when passing from Equation (27) to Equation (28). Therefore, the found solutions must be necessarily checked.

Check. For $x_1=\frac{1}{2}$ we get:

$$\arccos x_1-\arcsin x_1=\arccos\frac{1}{2}-\arcsin\frac{1}{2}=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}.$$

Thus, $x_1=\frac{1}{2}$ is a root of Equation (25).

For $x_2=-\frac{1}{2}$ we get:

$$\begin{aligned}\arccos x_2-\arcsin x_2&=\arccos\left(-\frac{1}{2}\right)\\ -\arcsin\left(-\frac{1}{2}\right)&=\frac{2\pi}{3}-\left(-\frac{\pi}{6}\right)\\ &=\frac{5\pi}{6}\neq\frac{\pi}{6}.\end{aligned}$$

Hence, $x_2=-\frac{1}{2}$ is an extraneous root. A check shows that the values $x_3=\frac{\sqrt{3}}{2}$ and $x_4=-\frac{\sqrt{3}}{2}$ are also extraneous. Thus, $x=\frac{1}{2}$.

Example 18. Solve the equation

$$\arcsin 2x+\arcsin x=\frac{\pi}{3}. \qquad (29)$$

Solution. Let us take the cosine of both sides of Equation (29):

$$\cos(\arcsin 2x+\arcsin x)=\cos\frac{\pi}{3},$$

then

$$\sqrt{1-4x^2}\sqrt{1-x^2}-2x\cdot x=\frac{1}{2},$$

whence $7x^2=\frac{3}{4}$, that is, $x_1=\frac{1}{2}\sqrt{\frac{3}{7}}$, $x_2=-\frac{1}{2}\sqrt{\frac{3}{7}}$.

Check. Set $\alpha=\arcsin 2x_1+\arcsin x_1$, then $\cos\left(\arcsin\sqrt{\frac{3}{7}}+\arcsin\left(\frac{1}{2}\sqrt{\frac{3}{7}}\right)\right)=\cos\alpha$, whence $\cos\alpha=\sqrt{1-\frac{3}{7}}\sqrt{1-\frac{1}{4}\times\frac{3}{7}}-\sqrt{\frac{3}{7}}\times\frac{1}{2}\times\sqrt{\frac{3}{7}}$, that is, $\cos\alpha=\frac{1}{2}$.

Since further $0<\sqrt{\frac{3}{7}}<\frac{\sqrt{2}}{2}$ and $0<\frac{1}{2}\sqrt{\frac{3}{7}}<\frac{\sqrt{2}}{2}$, we have: $0<\arcsin\sqrt{\frac{3}{7}}<\frac{\pi}{4}$ and $0<\arcsin\frac{1}{2}\sqrt{\frac{3}{7}}<\frac{\pi}{4}$.

Then $0<\arcsin\sqrt{\frac{3}{7}}+\arcsin\left(\frac{1}{2}\sqrt{\frac{3}{7}}\right)<\frac{\pi}{2}$, that is, α belongs to the first quadrant.

Thus, $\cos\alpha=\frac{1}{2}$ and $0<\alpha<\frac{\pi}{2}$, but in such a case $\alpha=\frac{\pi}{3}$, and, hence, $x_1=\frac{1}{2}\sqrt{\frac{3}{7}}$ is a root of Equation (29).

Let us now check the value $x_2=-\frac{1}{2}\sqrt{\frac{3}{7}}$. We set $\beta=\arcsin 2x_2+\arcsin x_2$, then $\arcsin\left(-\sqrt{\frac{3}{7}}\right)+\arcsin\left(-\frac{1}{2}\times\sqrt{\frac{3}{7}}\right)=\beta$. Since $-1<-\sqrt{\frac{3}{7}}<0$ and $-1<-\frac{1}{2}\sqrt{\frac{3}{7}}<0$, we have: $-\pi<\arcsin\left(-\sqrt{\frac{3}{7}}\right)+\arcsin\left(-\frac{1}{2}\sqrt{\frac{3}{7}}\right)<0$ or $-\pi<\beta<0$. Hence, $\beta\neq\frac{\pi}{3}$, whence it follows that $x_2=-\frac{1}{2}\sqrt{\frac{3}{7}}$ is an extraneous root.

Thus, Equation (29) has the only root $x=\frac{1}{2}\sqrt{\frac{3}{7}}$.

Example 19. Solve the equation

$$3\arcsin x+\pi x-\pi=0. \tag{30}$$

Solution. By trial and error method it is easy to find one root of the equation: $x_1=\frac{1}{2}$. Indeed,

$$3\arcsin\frac{1}{2}+\frac{\pi}{2}-\pi=3\times\frac{\pi}{6}-\frac{\pi}{2}=0.$$

Let us prove that there are no other roots. We transform the equation to

$$3\arcsin x=\pi-\pi x.$$

The function $y=3\arcsin x$ is increasing, while the function $y=\pi-\pi x$ is decreasing. If one side of an equation represents an increasing function, and the other a decreasing one, then the equation has either no root or only one root.

Thus, $x=\frac{1}{2}$ is the only root of Equation (30).

Remark. The method used for solving Equation (30) might be used for solving Equation (25) as well. Indeed, $y=\arccos x$ is a decreasing function, while $y=\frac{\pi}{6}+\arcsin x$ an increasing function, that

is, Equation (25) has either no root or only one root. By trial and error method, we find the only root of Equation (25): $x = \frac{1}{2}$.

Example 20. Solve the equation

$$\sin^4 x + \cos^4 y + 2 = 4 \sin x \cos y. \tag{31}$$

Solution. We get: $\begin{cases} u = \sin x \\ v = \cos y. \end{cases}$

Then Equation (31) will take the form:

$$u^4 + v^4 + 2 = 4uv. \tag{32}$$

Further, we have:

$$(u^4 + 1) + (v^4 + 1) - 4uv = 0,$$

$$(u^4 - 2u^2 + 1) + (v^4 - 2v^2 + 1) + 2u^2 + 2v^2 - 4uv = 0,$$

$$(u^2 - 1)^2 + (v^2 - 1)^2 + 2(u - v)^2 = 0. \tag{33}$$

Equation (33) is equivalent to the following system of equations:

$$\begin{cases} u^2 - 1 = 0 \\ v^2 - 1 = 0 \\ u - v = 0, \end{cases}$$

which is, in turn, equivalent to the collection of systems:

$$\begin{cases} u = 1 \\ v = 1 \\ u - v = 0 \end{cases}; \quad \begin{cases} u = 1 \\ v = -1 \\ u - v = 0 \end{cases}; \quad \begin{cases} u = -1 \\ v = 1 \\ u - v = 0 \end{cases}; \quad \begin{cases} u = -1 \\ v = -1 \\ u - v = 0. \end{cases}$$

The second and third systems of this collection have no solution and from the first and fourth systems we get, respectively:

$$\begin{cases} u_1 = 1 \\ v_1 = 1 \end{cases}; \quad \begin{cases} u_2 = -1 \\ v_2 = -1. \end{cases}$$

It remains to solve the collection of two systems of trigonometric equations:

$$\begin{cases} \sin x = 1 \\ \cos y = 1 \end{cases}; \quad \begin{cases} \sin x = -1 \\ \cos y = -1. \end{cases}$$

From this collection of systems we find:

$$\begin{cases} x_1 = \frac{\pi}{2} + 2\pi k \\ y_1 = 2\pi n \end{cases}; \quad \begin{cases} x_2 = -\frac{\pi}{2} + 2\pi k \\ y_2 = \pi + 2\pi n. \end{cases}$$

EXERCISES

In Problems 1432 through 1558, solve the given equations:

1432. $\frac{\cos x}{1+\cos 2x}=0.$ **1433.** $\frac{\sin x+\cos x}{\cos 2x}=0.$

1434. $\cos x \tan 3x=0.$ **1435.** $\sin 4x \cos x \tan 2x=0.$

1436. $(1+\cos x)\left(\frac{1}{\sin x}-1\right)=0.$ **1437.** $(1+\cos x)\tan\frac{x}{2}=0.$

1438. $\sin^2 3x-5\sin 3x+4=0.$ **1439.** $\tan^3 x+\tan^2 x-3\tan x=3.$

1440. $8\cos^4 x-8\cos^2 x-\cos x+1=0.$

1441. $2\sin^3 x-\cos 2x-\sin x=0.$ **1442.** $2\cos^2 x+5\sin x-4=0.$

1443. $3\sin^2 2x+7\cos 2x=3.$ **1444.** $2\cos^2 x+\sin x=2.$

1445. $\sqrt{2}\sin^2 x+\cos x=0.$ **1446.** $\sin 2x+\cos 2x=\sin x+\cos x.$

1447. $\sqrt{2}\cos 2x=\cos x+\sin x.$ **1448.** $\sin 3x=\cos 2x.$

1449. $\cos 5x=\sin 15x.$ **1450.** $\sin(5\pi-x)=\cos(2x+7\pi).$

1451. $4\sin^2 x+\sin^2 2x=3.$ **1452.** $4\cos^2 2x+8\cos^2 x=7.$

1453. $\sin\left(x+\frac{\pi}{6}\right)+\cos\left(x+\frac{\pi}{3}\right)=1+\cos 2x.$

1454. $8\sin^6 x+3\cos 2x+2\cos 4x+1=0.$

1455. $3(1-\sin x)=1+\cos 2x.$ **1456.** $\sin x=\frac{3}{4}\cos x.$

1457. $3\sin x=2\cos x.$ **1458.** $3\sin^2 x+3\sin x\cos x-6\cos^2 x=0.$

1459. $\sin^2 x+3\cos^2 x-2\sin 2x=0.$ **1460.** $3\sin^2 x+2\sin x\cos x=2.$

1461. $2\cos^2 x-3\sin x\cos x+5\sin^2 x=3.$

1462. $\sin 5x\cos 3x=\sin 9x\cos 7x.$ **1463.** $\sin 6x\cos 2x=\sin 5x\cos 3x-\sin 2x.$

1464. $\sin^6 x+\cos^6 x=\frac{7}{16}.$ **1465.** $2\cos^2 x+\cos 5x=1.$

1466. $\sin x+\sin 2x+\sin 3x=0.$ **1467.** $\sin x+\sin 3x+\cos x+\cos 3x=0.$

1468. $\sqrt{3}\sin 2x+\cos 2x=\sqrt{2}.$ **1469.** $\frac{1}{2}\sin 3x+\frac{\sqrt{3}}{2}\cos 3x=\sin 5x.$

1470. $2\cos 3x+\sqrt{3}\sin x+\cos x=0.$ **1471.** $\sin 5x+\cos 5x=\sqrt{2}\cos 13x.$

1472. $\sin^2 x-\cos 2x=2-\sin 2x.$

1473. $\sin^6 x+\sin^4 x\cos^2 x=\sin^2 x\cos^3 x+\sin x\cos^5 x.$

1474. $\sin^2 x\cos^2 x-10\sin x\cos^3 x+21\cos^4 x=0.$

1475. $8\sin^2\frac{x}{2}-3\sin x-4=0.$ **1476.** $\sin^4 x+\cos^4 x=\cos 4x.$

1477. $\cos^4 x+\sin^4 x-\sin 2x+\frac{3}{4}\sin^2 2x=0.$ **1478.** $3\tan\frac{x}{2}+\cot x=\frac{5}{\sin x}.$

1479. $\cos 2x-3\cos x+1=\frac{1}{(\cot 2x-\cot x)\sin(x-\pi)}.$

1480. $\cos x=\frac{\tan x}{1+\tan^2 x}.$ **1481.** $\cot x+\frac{\sin x}{1+\cos x}=2.$

1482. $2\sin x-3\cos x=3.$ **1483.** $3\sin 2x+\cos 2x=2.$

1484. $\cos 4x+2\sin 4x=1$. **1485.** $\sin 2x+\tan x=2$.

1486. $\dfrac{1+\sin x}{1+\cos x}=\dfrac{1}{2}$. **1487.** $\sin^3 x+\cos^3 x=1$.

1488. $4\sin^4 3x-3\cos x+5=0$.

1489. $\sin x\cos x-6\sin x+6\cos x+6=0$.

1490. $4-4(\cos x-\sin x)-\sin 2x=0$.

1491. $5\sin 2x-11(\sin x+\cos x)+7=0$.

1492. $\left(2\sin^4\dfrac{x}{2}-1\right)\dfrac{1}{\cos^4\dfrac{x}{2}}=2$. **1493.** $\cos x\cos 2x\cos 4x\cos 8x=\dfrac{1}{16}$.

1494. $2\sin 17x+\sqrt{3}\cos 5x+\sin 5x=0$.

1495. $4\cos^3\dfrac{x}{2}+3\sqrt{2}\sin x=8\cos\dfrac{x}{2}$.

1496. $\dfrac{7}{4}\cos\dfrac{x}{4}=\cos^3\dfrac{x}{4}+\sin\dfrac{x}{2}$. **1497.** $4\sin 2x-\tan^2\left(x-\dfrac{\pi}{4}\right)=4$.

1498. $(\sin 2x+\sqrt{3}\cos 2x)^2-5=\cos\left(\dfrac{\pi}{6}-2x\right)$.

1499. $\dfrac{1-\sin x+\ldots+(-1)^n\sin^n x+\ldots}{1+\sin x+\ldots+\sin^n x+\ldots}=\dfrac{1-\cos 2x}{1+\cos 2x}$.

1500. $\cos\dfrac{4x}{3}=\cos^2 x$. **1501.** $\sin x+2\cos x=\cos 2x-\sin 2x$.

1502. $32\cos^6 x-\cos 6x=1$. **1503.** $\tan x+\cot x-\cos 4x=3$.

1504. $2(1-\sin x-\cos x)+\tan x+\cot x=0$.

1505. $\sin^5 x-\cos^5 x=\dfrac{1}{\cos x}-\dfrac{1}{\sin x}$.

1506. $\sin^8 2x+\cos^8 2x=\dfrac{41}{128}$. **1507.** $\sin^{10} x+\cos^{10} x=\dfrac{29}{64}$.

1508. $\sin^{10} x+\cos^{10} x=\dfrac{29}{16}\cos^4 2x$. **1509.** $|\cos x|=\cos x-2\sin x$.

1510. $|\cot x|=\cot x+\dfrac{1}{\sin x}$. **1511.** $\sqrt{5-2\sin x}=6\sin x-1$.

1512. $\sqrt{2+4\cos x}=\dfrac{1}{2}+3\cos x$. **1513.** $\sqrt{3+2\tan x-\tan^2 x}=\dfrac{1+3\tan x}{2}$.

1514. $\sqrt{-3\sin 5x-\cos^2 x-3}+\sin x=1$.

1515. $\tan x+\dfrac{1}{9}\cot x=\sqrt{\dfrac{1}{\cos^2 x}-1}-1$.

1516. $(1+\cos x)\sqrt{\tan\dfrac{x}{2}}-2+\sin x=2\cos x$.

1517. $\sqrt{\cos^2 x+\dfrac{1}{2}}+\sqrt{\sin^2 x+\dfrac{1}{2}}=2$.

1518. $\sqrt{1-2\tan x}-\sqrt{1+2\cot x}=2$.

1519. $\sqrt{3}\sin x-\sqrt{2\sin^2 x-\sin 2x+3\cos^2 x}=0$.

1520. $\cos x+\sqrt{\sin^2 x-2\sin 2x+4\cos^2 x}=0$.

1521. $\sqrt{\cos 2x}+\sqrt{1+\sin 2x}=2\sqrt{\sin x+\cos x}$.

1522. $2\cot 2x-3\cot 3x=\tan 2x$.

1523. $6\tan x+5\cot 3x=\tan 2x$.

1524. $\tan\left(x-\frac{\pi}{4}\right)\tan x\tan\left(x+\frac{\pi}{4}\right)=\dfrac{4\cos^2 x}{\tan\frac{x}{2}-\cot\frac{x}{2}}$.

1525. $\sin^2 5x\left(\sin 7x\cos x-\sin\frac{x}{2}\cos 7x\right)=\dfrac{\sin\frac{3x}{2}\cos\frac{x}{2}+\sin x\cdot\cos 7x}{1+\cot^2 5x}$.

1526. $\sin^6 x+\sin^4 x+\cos^6 x+\cos^4 x+\sin\frac{x}{2}=3$.

1527. $1+\cos 2x\cos 3x=\frac{1}{2}\sin^2 3x$.

1528. $\sin 5x+\sin x=2+\cos^2 x$. **1529.** $3\sin^2\frac{x}{3}+5\sin^2 x=8$.

1530. $(\sin x+\sqrt{3}\cos x)\sin 3x=2$.

1531. $2\sin\left(\frac{2}{3}x-\frac{\pi}{6}\right)-3\cos\left(2x+\frac{\pi}{3}\right)=5$.

1532. $\sin\frac{x}{4}+2\cos\frac{x-2\pi}{3}=3$.

1533. $\sin 18x+\sin 10x+\sin 2x=3+\cos^2 2x$.

1534. $\cos 2x\left(1-\frac{3}{4}\sin^2 2x\right)=1$. **1535.** $4x^4+x^6=-\sin^2 5x$.

536. $\sin x+\cos x=\sqrt{2}+\sin^4 4x$.

1537. $\cos^6 2x=1+\sin^4 x$. **1538.** $\cot\left(\frac{\pi}{3}\cos 2\pi x\right)=\sqrt{3}$.

1539. $2\sin^2\left(\frac{\pi}{2}\cos^2 x\right)=1-\cos(\pi\sin 2x)$.

1540. $4\arctan(x^2-3x+3)=\frac{\pi}{4}$. **1541.** $\arctan 3x-\operatorname{arccot} 3x=\frac{\pi}{4}$.

1542. $2\arcsin^2 x-5\arcsin x+2=0$. **1543.** $4\arctan x-6\operatorname{arccot} x=\pi$.

1544. $\arcsin x+\arccos(1-x)=\arcsin(-x)$. **1545.** $2\arcsin x=\arccos 2x$.

1546. $\arcsin\frac{x}{2}+\arccos\left(x+\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$. **1547.** $\arccos x=\arctan x$.

1548. $\arcsin\dfrac{2}{3\sqrt{x}}-\arcsin\sqrt{1-x}=\arcsin\frac{1}{3}$.

1549. $3\arccos x-\pi x-\frac{\pi}{2}=0$.

1550. $\arcsin\left(\tan\frac{\pi}{4}\right)-\arcsin\sqrt{\frac{3}{x}}-\frac{\pi}{6}=0$.

1551. $\cos(x-y)-2\sin x+2\sin y=3$.

1552. $\sin^2(\pi x)+\log_2^2(y^2-2y+1)=0$.

1553. $\left(\sin^2 x+\dfrac{1}{\sin^2 x}\right)+\left(\cos^2 x+\dfrac{1}{\cos^2 x}\right)^2=12+\frac{1}{2}\sin y$.

1554. $1-2x-x^2=\tan^2(x+y)+\cot^2(x+y)$.

1555. $\tan^2 2x+2\sqrt{3}\tan 2x+3=-\cot^2\left(4y-\frac{\pi}{6}\right).$

1556. $x^2+2x\sin xy+1=0.$ **1557.** $4+\sin^2 x+\cos^2 2x=5\sin^2 x\sin^2 y.$

1558. $\left(\cos^2 x+\frac{1}{\cos^2 x}\right)(1+\tan^2 2y)(3+\sin 3z)=4.$

1559. Solve the equation $\arctan x+\arctan\frac{1}{y}=\arctan 3$ in integers.

SEC. 25. SYSTEMS OF EQUATIONS

When solving systems of trigonometric and algebraic equations, we use the same methods. Instead of general formulas for solving equations of the form $\sin x=a$, $\cos x=a$ it is often reasonable to write the solutions of these equations as a collection of two families. For instance, let us solve the system of equations

$$\begin{cases}\sin(x+y)=\frac{1}{2}\\ \cos(x-y)=\frac{\sqrt{2}}{2}.\end{cases}\tag{1}$$

Using the general formulas, we get the system:

$$\begin{cases}x+y=(-1)^k\frac{\pi}{6}+\pi k\\ x-y=\pm\frac{\pi}{4}+2\pi n,\end{cases}\tag{2}$$

whence we find:

$$\begin{cases}x_{1,2}=(-1)^k\frac{\pi}{12}\pm\frac{\pi}{8}+\frac{\pi k}{2}+\pi n\\ y_{1,2}=(-1)^k\frac{\pi}{12}\mp\frac{\pi}{8}+\frac{\pi k}{2}-\pi n\end{cases}\tag{3}$$

which is the solution of System (1). If the solution of the first equation of System (1) is written as the collection $x+y=\frac{\pi}{6}+2\pi k$; $x+y=\frac{5\pi}{6}+2\pi k$; and the solution of the second equation as the collection $x-y=\frac{\pi}{4}+2\pi n$; $x-y=-\frac{\pi}{4}+2\pi n$, we get the collection of four systems:

$$\begin{cases}x+y=\frac{\pi}{6}+2\pi k\\ x-y=\frac{\pi}{4}+2\pi n\end{cases};\qquad\begin{cases}x+y=\frac{5\pi}{6}+2\pi k\\ x-y=\frac{\pi}{4}+2\pi n\end{cases};\tag{4}$$

$$\begin{cases} x+y=\frac{\pi}{6}+2\pi k \\ x-y=-\frac{\pi}{4}+2\pi n \end{cases}; \quad \begin{cases} x+y=\frac{5\pi}{6}+2\pi k \\ x-y=-\frac{\pi}{4}+2\pi n, \end{cases}$$

whence

$$\begin{cases} x_1=\frac{5\pi}{24}+\pi(k+n) \\ y_1=-\frac{\pi}{24}+\pi(k-n) \end{cases}; \quad \begin{cases} x_2=\frac{13\pi}{24}+\pi(k+n) \\ y_2=\frac{7\pi}{24}+\pi(k-n) \end{cases};$$

$$\begin{cases} x_3=-\frac{\pi}{24}+\pi(k+n) \\ y_3=\frac{5\pi}{24}+\pi(k-n) \end{cases}; \quad \begin{cases} x_4=\frac{7\pi}{24}+\pi(k+n) \\ y_4=\frac{13\pi}{24}+\pi(k-n). \end{cases}$$

This collection of families represents the solution of System (1). Of course, such representation is not so compact as System (3), but more clear and therefore preferable in many cases.

We should like to draw the reader's attention to the following point: when passing from System (1) to System (2) or to the collection of Systems (4), we used the parameter k for representing the solutions of the first equation of System (1), and the parameter n for representing the solutions of the second equation of the system. The use of only one parameter, say k, would lead us to a loss of solutions: in this case from the first system of Collection (4) we would get: $\begin{cases} x_1'=\frac{5\pi}{24}+2\pi k \\ y_1'=-\frac{\pi}{24}, \end{cases}$ the set Z_1' of pairs (x_1', y_1') representing a proper subset of the set Z_1 of pairs (x_1, y_1), where $\begin{cases} x_1=\frac{5\pi}{24}+\pi(k+n) \\ y_1=-\frac{\pi}{24}+\pi(k-n) \end{cases}$. Thus, $Z' \in Z_1$, $Z' \neq Z_1$, therefore all pairs (x, y) such that $(x, y) \in Z_1 \setminus Z'$ turn out to be "lost" solutions.

Let us consider several examples.

Example 1. Solve the system of equations

$$\begin{cases} \sin x \sin y = 0.75 \\ \tan x \tan y = 3. \end{cases} \tag{5}$$

Solution. Dividing the left-hand and right-hand sides of the first equation of System (5) respectively by the left-hand and right-hand sides of the second equation of the system, we get the equation: $\cos x \cos y = \frac{1}{4}$. Replacing the second equation of System (5) by this

equation, we get the system:

$$\begin{cases} \sin x \sin y = \frac{3}{4} \\ \cos x \cos y = \frac{1}{4}, \end{cases} \tag{6}$$

which is equivalent to System (5).

Let us now replace the first equation of System (6) by the sum of the equations of this system, and the second equation by the difference between the second and first equations. We thus get a new system:

$$\begin{cases} \cos x \cos y + \sin x \sin y = 1 \\ \cos x \cos y - \sin x \sin y = -\frac{1}{2} \end{cases} \tag{7}$$

or

$$\begin{cases} \cos (x - y) = 1 \\ \cos (x + y) = -\frac{1}{2}, \end{cases}$$

which is equivalent to System (6). From the first equation of System (7) we find: $x - y = 2\pi k$. The second equation of System (7) is equivalent to the collection of equations $x + y = \frac{2\pi}{3} + 2\pi n$; $x + y = -\frac{2\pi}{3} + 2\pi n$.

Thus, from System (7) we have passed to the collection of systems

$$\begin{cases} x - y = 2\pi k \\ x + y = \frac{2\pi}{3} + 2\pi n \end{cases}; \quad \begin{cases} x - y = 2\pi k \\ x + y = -\frac{2\pi}{3} + 2\pi n, \end{cases} \tag{8}$$

which is equivalent to System (7). From the first system of Collection (8) we find the family of solutions:

$$\begin{cases} x_1 = \frac{\pi}{3} + \pi (n + k) \\ y_1 = \frac{\pi}{3} + \pi (n - k). \end{cases}$$

From the second system of Collection (8) we find the family:

$$\begin{cases} x_2 = -\frac{\pi}{3} + \pi (n + k) \\ y_2 = -\frac{\pi}{3} + \pi (n - k). \end{cases}$$

Check. Since in the process of solution we carried out only equivalent transformations (this fact has been noted), the collection of

families

$$\begin{cases} x_1 = \frac{\pi}{3} + \pi(n+k) \\ y_1 = \frac{\pi}{3} + \pi(n-k) \end{cases}; \quad \begin{cases} x_2 = -\frac{\pi}{3} + \pi(n+k) \\ y_2 = -\frac{\pi}{3} + \pi(n-k) \end{cases}$$

is the solution of System (5).

Example 2. Solve the system of equations

$$\begin{cases} \sin^3 x = \frac{1}{2} \sin y \\ \cos^3 x = \frac{1}{2} \cos y. \end{cases} \tag{9}$$

Solution. We square both sides of each equation of System (9) and add termwise the equations yielded by this transformation. We shall have: $\sin^6 x + \cos^6 x = \frac{1}{4}$, and from System (9) we shall pass to a new system:

$$\begin{cases} \sin^6 x + \cos^6 x = \frac{1}{4} \\ \sin^3 x = \frac{1}{2} \sin y. \end{cases} \tag{10}$$

Solving the equation $\sin^6 x + \cos^6 x = \frac{1}{4}$, we get:

$$\left(\frac{1-\cos 2x}{2}\right)^3 + \left(\frac{1+\cos 2x}{2}\right)^3 = \frac{1}{4}, \quad \cos 2x = 0, \quad x = \frac{\pi}{4} + \frac{\pi}{2} k.$$

Thus, the solution of System (10) has been reduced to solving the system

$$\begin{cases} x = \frac{\pi}{4} + \frac{\pi}{2} k \\ \sin^3 x = \frac{1}{2} \sin y. \end{cases} \tag{11}$$

We have: $\begin{cases} x = \frac{\pi}{4} + \frac{\pi}{2} k \\ \frac{1}{2} \sin y = \left(\pm \frac{\sqrt{2}}{2}\right)^3 \end{cases}$ or $\begin{cases} x = \frac{\pi}{4} + \frac{\pi}{2} k \\ \sin y = \pm \frac{\sqrt{2}}{2} \end{cases}$, whence

$$\begin{cases} x = \frac{\pi}{4} + \frac{\pi}{2} k \\ y = \frac{\pi}{4} + \frac{\pi}{2} n. \end{cases} \tag{12}$$

The passage from System (9) to System (10) was, not possibly, an equivalent transformation (squaring), therefore a check is needed.

Check. Let us represent the values of x and y contained in System (12) by points of two circles (Fig. 38). At point A_1 we have: $\sin x > 0$,

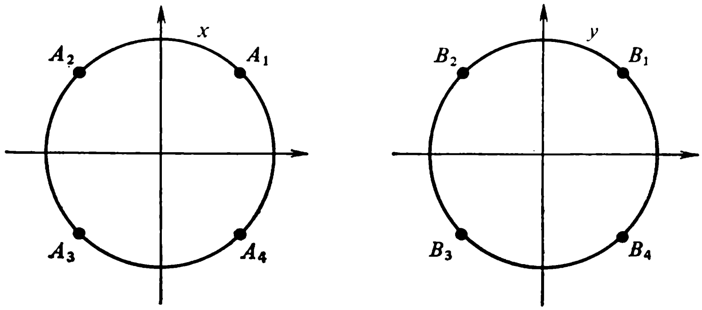

Fig. 38

$\cos x > 0$. Then from System (9) we conclude that $\sin y > 0$ and $\cos y > 0$. But of points B_1, B_2, B_3, B_4 only the point B_1 has a positive abscissa and ordinate. Hence, (A_1, B_1) is a geometric solution of System (9), that is,

$$\begin{cases} x_1 = \frac{\pi}{4} + 2\pi k \\ y_1 = \frac{\pi}{4} + 2\pi n \end{cases}$$

is the solution of System (9).

Reasoning in a similar way, we get: (A_2, B_2), (A_3, B_3), (A_4, B_4) which are geometric solutions of System (9), that is,

$$\begin{cases} x_2 = \frac{3\pi}{4} + 2\pi k \\ y_2 = \frac{3\pi}{4} + 2\pi n \end{cases}; \quad \begin{cases} x_3 = \frac{5\pi}{4} + 2\pi k \\ y_3 = \frac{5\pi}{4} + 2\pi n \end{cases}; \quad \begin{cases} x_4 = \frac{7\pi}{4} + 2\pi k \\ y_4 = \frac{7\pi}{4} + 2\pi n \end{cases}$$

are solutions of System (9).

Thus, the solution of System (9) is represented by the following collection of families:

$$\begin{cases} x_1 = \frac{\pi}{4} + 2\pi k \\ y_1 = \frac{\pi}{4} + 2\pi n \end{cases}; \quad \begin{cases} x_2 = \frac{3\pi}{4} + 2\pi k \\ y_2 = \frac{3\pi}{4} + 2\pi n \end{cases};$$

$$\begin{cases} x_3 = \frac{5\pi}{4} + 2\pi k \\ y_3 = \frac{5\pi}{4} + 2\pi n \end{cases}; \quad \begin{cases} x_4 = \frac{7\pi}{4} + 2\pi k \\ y_4 = \frac{5\pi}{4} + 2\pi n. \end{cases}$$

Example 3. Solve the system of equations

$$\begin{cases} x+y+z=\pi \\ \tan x \tan z = 2 \\ \tan y \tan z = 18. \end{cases} \tag{13}$$

Solution. Since $x + y + z = \pi$, we have: $\tan(x + y) = \tan(\pi - z)$, that is, $\frac{\tan x + \tan y}{1 - \tan x \tan y} = -\tan z$.

Let us replace by this equation the first equation of System (13) and consider the new system:

$$\begin{cases} \frac{\tan x + \tan y}{1 - \tan x \tan y} = -\tan z \\ \tan x \tan z = 2 \\ \tan y \tan z = 18. \end{cases} \tag{14}$$

We then introduce new variables:

$$\begin{cases} u = \tan x \\ v = \tan y \\ w = \tan z. \end{cases}$$

Then System (14) will take the form:

$$\begin{cases} \frac{u+v}{1-uv} = -w \\ uw = 2 \\ vw = 18, \end{cases} \tag{15}$$

or

$$\begin{cases} uvw = u+v+w \\ uw = 2 \\ vw = 18. \end{cases} \tag{16}$$

Dividing termwise the first equation of System (16) by the second, we get: $v = \frac{u+v+w}{2}$, whence $v = u + w$. Replacing by this equation the first equation of System (16), we get:

$$\begin{cases} v = u+w \\ uw = 2 \\ vw = 18, \end{cases} \tag{17}$$

and further

$$\begin{cases} v = u+w \\ uw = 2 \\ (u+w)w = 18, \end{cases} \qquad \begin{cases} v = u+w \\ uw = 2 \\ w^2 = 16. \end{cases} \tag{18}$$

System (18) has the following solutions:

$$\begin{cases} u_1 = 0.5 \\ v_1 = 4.5 \\ w_1 = 4 \end{cases}; \quad \begin{cases} u_2 = -0.5 \\ v_2 = -4.5 \\ w_2 = -4. \end{cases}$$

Returning to the original variables, we get:

$$\begin{cases} x_1 = \arctan 0.5 + \pi k \\ y_1 = \arctan 4.5 + \pi n \\ z_1 = \arctan 4 + \pi m \end{cases}; \quad \begin{cases} x_2 = -\arctan 0.5 + \pi k \\ y_2 = -\arctan 4.5 + \pi n \\ z_2 = -\arctan 4 + \pi m. \end{cases} \qquad (19)$$

Check. In the process of solution, there were three transformations each of which could lead to a non-equivalent system: (1) "taking tangent" when passing from System (13) to System (14), (2) getting rid of the denominator when passing from System (15) to System (16), and (3) the division when passing from System (16) to System (17). Only division might lead to a loss of solutions, but this was not the case in our problem since the right-hand side of the "divisor equation" is 2, that is, it is different from zero. The remaining transformations might lead to extraneous solutions which can be rejected by the direct substitution of the values contained in the above collection (19) into the original system. It is easy to make sure that Collection (19) satisfies the second and third equations of System (13). In order for the first equation of this system to be satisfied, we have to write the solutions of Collection (19) as follows:

$$\begin{cases} x_1 = \arctan 0.5 + \pi k \\ y_1 = \arctan 4.5 + \pi n \\ z_1 = \arctan 4 - \pi k - \pi n \end{cases}; \quad \begin{cases} x_2 = -\arctan 0.5 + \pi k \\ y_2 = -\arctan 4.5 + \pi n \\ z_2 = -\arctan 4 - \pi k - \pi n + 2\pi \end{cases} \qquad (20)$$

(as follows from the solution of the example, $\arctan 0.5 + \arctan 4.5 + \arctan 4 = \pi$).

Collection of families (20) represents the solution of System (13).

Example 4. Solve the system of equations

$$\begin{cases} \sin x = \cos y \\ \sqrt{6} \sin y = \tan z \\ 2 \sin z = \sqrt{3} \cot x. \end{cases} \qquad (21)$$

Solution. Squaring both sides of each of the equations of System (21), we get:

$$\begin{cases} \sin^2 x = \cos^2 y \\ 6 \sin^2 y = \tan^2 z \\ 4 \sin^2 z = 3 \cot^2 x. \end{cases} \qquad (22)$$

Let us introduce new variables: $\begin{cases} u = \sin^2 x \\ v = \sin^2 y \\ w = \sin^2 z. \end{cases}$

Then System (22) will take the form:

$$\begin{cases} u = 1 - v \\ 6v = \dfrac{w}{1-w} \\ 4w = 3\,\dfrac{1-u}{u}, \end{cases} \tag{23}$$

whence we find:

$$\begin{cases} u_1 = 1 \\ v_1 = 0 \\ w_1 = 0 \end{cases} ; \quad \begin{cases} u_2 = \dfrac{1}{2} \\ v_2 = \dfrac{1}{2} \\ w_2 = \dfrac{3}{4} \end{cases}$$

which are solutions of System (23).

Now, the problem has been reduced to solving the following collection of systems:

$$\begin{cases} \sin^2 x = 1 \\ \sin^2 y = 0 \\ \sin^2 z = 0 \end{cases} ; \quad \begin{cases} \sin^2 x = \dfrac{1}{2} \\ \sin^2 y = \dfrac{1}{2} \\ \sin^2 z = \dfrac{3}{4}. \end{cases} \tag{24}$$

From the first system of this collection we find:

$$\begin{cases} x = \dfrac{\pi}{2} + \pi k \\ y = \pi n \\ z = \pi m. \end{cases} \tag{25}$$

From the second system of Collection (24) we find:

$$\begin{cases} x = \dfrac{\pi}{4} + \dfrac{\pi k}{2} \\ y = \dfrac{\pi}{4} + \dfrac{\pi n}{2} \\ z = \pm\dfrac{\pi}{3} + \pi m. \end{cases} \tag{26}$$

Check. Let us substitute the found solutions (25) and (26) into the original system (21). To this end, let us represent x, y, z from System (25) by points of three circles (Fig. 39) as it has been done in Example 3.

Let us take point A_1. At this point $\sin x > 0$, and therefore $\cos y > 0$ (see the first equation of System (21)). Then, of two points

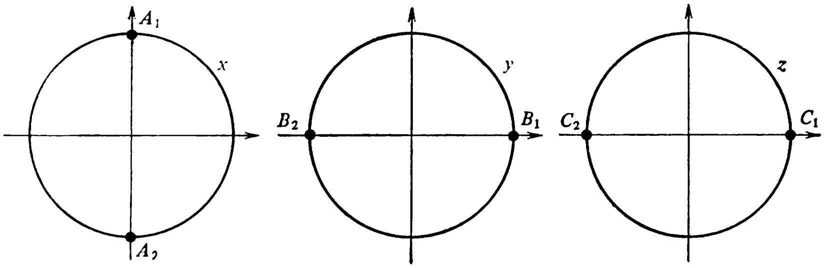

Fig. 39

B_1, B_2 we choose the point with a positive abscissa, that is, B_1. Note that in this case either of points C_1, C_2 may be taken. Similarly, point B_2 corresponds to point A_2. Thus, we have obtained four

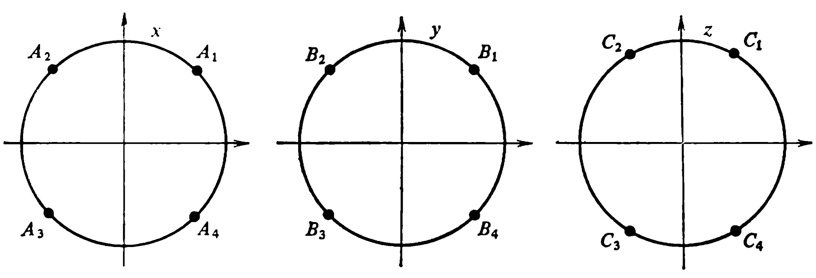

Fig. 40

geometric solutions: (A_1, B_1, C_1), (A_1, B_1, C_2), (A_2, B_2, C_1), (A_2, B_2, C_2).

Thus, instead of Family (25), we get the following collection of families:

$$\begin{cases} x_1 = \frac{\pi}{2} + 2\pi k \\ y_1 = 2\pi n \\ z_1 = \pi m \end{cases} ; \quad \begin{cases} x_2 = -\frac{\pi}{2} + 2\pi k \\ y_2 = \pi + 2\pi n \\ z_2 = \pi m \end{cases} \tag{27}$$

(the rest of the suites (x, y, z) contained in Family (25) are extraneous solutions for the original system).

Let us now represent x, y, z from System (26) (Fig. 40) by points on the circles. Consider point A_1. At this point $\sin x > 0$, $\cot x > 0$,

hence, $\cos y > 0$, $\sin z > 0$ (see the first and third equations of System (21)). Since $\cos y > 0$, on the second circle we choose points with positive abscissas: B_1 and B_4. Since $\sin z > 0$, on the third circle we choose points with positive ordinates: C_1 and C_2. Consider point B_1. At this point $\sin y > 0$, hence, $\tan z > 0$ (see the second equation of System (21)), and therefore of points C_1, C_2 we choose point C_1 (at which $\tan z > 0$). Similarly, point C_2 will correspond to point B_2.

Thus, we have obtained two more geometric solutions: (A_1, B_1, C_1) and (A_1, B_2, C_2) and, respectively, the following collection of families of solutions of System (21):

$$\begin{cases} x_3=\frac{\pi}{4}+2\pi k \\ y_3=\frac{\pi}{4}+2\pi n \\ z_3=\frac{\pi}{4}+2\pi m \end{cases}; \quad \begin{cases} x_4=\frac{\pi}{4}+2\pi k \\ y_4=\frac{3\pi}{4}+2\pi n \\ z_4=\frac{2\pi}{3}+2\pi m. \end{cases} \tag{28}$$

Reasoning in a similar way, we find six more geometric solutions: (A_2, B_1, C_3), (A_2, B_4, C_4), (A_3, B_2, C_1), (A_3, B_3, C_2), (A_4, B_2, C_3), (A_4, B_3, C_4) and the collection of respective families of solutions:

$$\begin{cases} x_5=\frac{3\pi}{4}+2\pi k \\ y_5=\frac{\pi}{4}+2\pi n \\ z_5=\frac{4\pi}{3}+2\pi m \end{cases}; \quad \begin{cases} x_6=\frac{3\pi}{4}+2\pi k \\ y_6=\frac{7\pi}{4}+2\pi n \\ z_6=\frac{5\pi}{3}+2\pi m \end{cases}; \quad \begin{cases} x_7=\frac{5\pi}{4}+2\pi k \\ y_7=\frac{3\pi}{4}+2\pi n \\ z_7=\frac{\pi}{3}+2\pi m \end{cases};$$
$$\begin{cases} x_8=\frac{5\pi}{4}+2\pi k \\ y_8=\frac{5\pi}{4}+2\pi n \\ z_8=\frac{2\pi}{3}+2\pi m \end{cases}; \quad \begin{cases} x_9=\frac{7\pi}{4}+2\pi k \\ y_9=\frac{3\pi}{4}+2\pi n \\ z_9=\frac{4\pi}{3}+2\pi m \end{cases}; \quad \begin{cases} x_{10}=\frac{7\pi}{4}+2\pi k \\ y_{10}=\frac{5\pi}{4}+2\pi n \\ z_{10}=\frac{5\pi}{3}+2\pi m. \end{cases} \tag{29}$$

Thus, the collection of families (27), (28), and (29) is the solution of System (21).

EXERCISES

In Problems 1560 through 1602, solve the given systems of equations:

1560. $\begin{cases} \sin(x+y)=0 \\ \sin(x-y)=0. \end{cases}$ **1561.** $\begin{cases} \sin x \cos y=0.25 \\ \sin y \cos x=0.75. \end{cases}$

1562. $\begin{cases}\sin x+\cos y=0\\ \sin^2 x+\cos^2 y=\frac{1}{2}.\end{cases}$

1563. $\begin{cases}\sin x\sin y=0.25\\ x+y=\frac{\pi}{3}.\end{cases}$

1564. $\begin{cases}\sin x+\frac{1}{\cos y}=2\\ \frac{\sin x}{\cos y}=0.5.\end{cases}$

1565. $\begin{cases}\cos x+\cos y=0.5\\ \sin^2 x+\sin^2 y=1.75.\end{cases}$

1566. $\begin{cases}\sin x+\sin y=0\\ \cos x+\cos y=0.\end{cases}$

1567. $\begin{cases}x-y=\frac{1}{3}\\ \cos^2 \pi x-\sin^2 \pi y=\frac{1}{2}.\end{cases}$

1568. $\begin{cases}\sin^2 x+\sin^2 y=\frac{3}{4}\\ x+y=\frac{\pi}{3}.\end{cases}$

1569. $\begin{cases}\cos^2 x+\cos^2 y=0.25\\ x+y=\frac{5\pi}{6}.\end{cases}$

1570. $\begin{cases}\sin^2 x+\cos^2 y=\frac{1}{2}\\ x+y=\frac{\pi}{4}.\end{cases}$

1571. $\begin{cases}\cos x+\cos y=1\\ \cos\frac{x}{2}+\cos\frac{y}{2}=\frac{\sqrt{2}-2}{2}.\end{cases}$

1572. $\begin{cases}\cos x\sin y=\frac{\sqrt{2}}{2}\\ x+y=\frac{3\pi}{4}.\end{cases}$

1573. $\begin{cases}\frac{1-\tan x}{1+\tan x}=\tan y\\ x-y=\frac{\pi}{6}.\end{cases}$

1574. $\begin{cases}\tan x+\tan y=1\\ x+y=\frac{\pi}{3}.\end{cases}$

1575. $\begin{cases}\sin x\cot y=\frac{\sqrt{6}}{2}\\ \tan x\cos y=\frac{\sqrt{3}}{2}.\end{cases}$

1576. $\begin{cases}\cos(x-y)=2\cos(x+y)\\ \cos x\cos y=0.75.\end{cases}$

1577. $\begin{cases}\sin(x-y)=3\sin x\cos y-1\\ \sin(x+y)=-2\cos x\sin y.\end{cases}$

1578. $\begin{cases}\cos\frac{x+y}{2}\cos\frac{x-y}{2}=\frac{1}{2}\\ \cos x\cos y=\frac{1}{4}.\end{cases}$

1579. $\begin{cases}\sin x\sin y=\frac{1}{4\sqrt{2}}\\ \tan x\tan y=\frac{1}{3}.\end{cases}$

1580. $\begin{cases}\sin x=3\sin y\\ \tan x=5\tan y.\end{cases}$

1581. $\begin{cases}x+y=\frac{\pi}{4}\\ \frac{\tan x}{\tan y}=\frac{3}{4}.\end{cases}$

1582. $\begin{cases}x+y=\frac{\pi}{6}\\ 5(\sin 2x+\sin 2y)=2(1+\cos^2(x-y)).\end{cases}$

1583. $\begin{cases}x-y=\frac{5\pi}{3}\\ \sin x=2\sin y.\end{cases}$

1584. $\begin{cases}x+y=\frac{\pi}{4}\\ \tan x\tan y=\frac{1}{6}.\end{cases}$

1585. $\begin{cases}\sqrt{2}\sin x=\sin y\\ \sqrt{2}\cos x=\sqrt{3}\cos y.\end{cases}$

1586. $\begin{cases} \sin x \cos (x+y)+\sin (x+y)=3 \cos (x+y) \\ 4 \sin x=5 \cot (x+y). \end{cases}$

1587. $\begin{cases} \cot x+\sin 2y=\sin 2x \\ 2 \sin y \sin (x+y)=\cos x. \end{cases}$ 1588. $\begin{cases} 4 \tan 3y=3 \tan 2x \\ 2 \sin x \cos (x-y)=\sin y. \end{cases}$

1589. $\begin{cases} \tan x+\cot y=3 \\ |x-y|=\dfrac{\pi}{3}. \end{cases}$ 1590. $\begin{cases} \sin x=\sin 2y \\ \cos x=\sin y. \end{cases}$

1591. $\begin{cases} x+y=\dfrac{2\pi}{3} \\ \dfrac{\sin x}{\sin y}=2. \end{cases}$ 1592. $\begin{cases} \sin x-\sin y=\dfrac{1}{2} \\ \cos x+\cos y=\dfrac{\sqrt{3}}{2}. \end{cases}$

1593. $\begin{cases} \sin y=5 \sin x \\ 3 \cos x+\cos y=2. \end{cases}$ 1594. $\begin{cases} \cos x \cos y=\dfrac{1+\sqrt{2}}{4} \\ \cot x \cot y=3+2\sqrt{2}. \end{cases}$

1595. $\begin{cases} \sin^2 x=\cos x \cos y \\ \cos^2 x=\sin x \sin y. \end{cases}$ 1596. $\begin{cases} \cos^2 y+3 \sin x \sin y=0 \\ 21 \cos 2x-\cos 2y=10. \end{cases}$

1597. $\begin{cases} \cos^2 4x+\dfrac{\sqrt{26}-2}{2} \tan (-2y)=\dfrac{\sqrt{26}-1}{4} \\ \tan^2 (-2y)-\dfrac{\sqrt{26}-2}{2} \cos 4x=\dfrac{\sqrt{26}-1}{4}. \end{cases}$

1598. $\begin{cases} \sin^2 x=\sin y \\ \cos^4 x=\cos y. \end{cases}$ 1599. $\begin{cases} x+y+z=\pi \\ \tan x \tan z=3 \\ \tan y \tan z=6. \end{cases}$

1600. $\begin{cases} x+y+z=\pi \\ \tan x \tan y=2 \\ \tan x+\tan y+\tan z=6. \end{cases}$ 1601. $\begin{cases} x+y+z=\pi \\ \sin x=2 \sin y \\ \sqrt{3} \sin y=\sin z. \end{cases}$

1602. $\begin{cases} \sin^2 x+\sin^2 y+\sin^2 z=1 \\ \cos^2 x+\cos^2 y-\cos^2 z=1 \\ \tan^2 x-\tan^2 y+\tan^2 z=1. \end{cases}$

1603. Find the solutions of the system of equations

$$\begin{cases} |\sin x| \sin y=-\dfrac{1}{4} \\ \cos (x+y)+\cos (x-y)=\dfrac{3}{2} \end{cases}$$

Satisfying the conditions: $\begin{cases} 0<x<2\pi \\ \pi<y<2\pi. \end{cases}$

SEC. 26. INEQUALITIES

The solution of trigonometric inequalities is reduced, as a rule, to solving simplest trigonometric inequalities, that is, inequalities of the form $\sin x > a$, $\cos x < a$, etc., and also to solving collec-

tions, systems or collections of systems of simplest trigonometric inequalities. For solving them it is convenient sometimes to use a circle on which the set of values of the variable satisfying a given inequality is represented by one or several arcs.

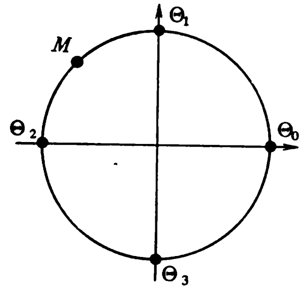

Fig. 41

By analogy with intervals given by inequalities on the number line, it is possible to represent sets of points belonging to one or other arc of the circle.

Let us agree to use the symbol $\cup M_1M_2$ for denoting an arc, where M_1 is its initial point and M_2 the terminate point of the path described by a point moving in the circle Σ in the positive direction (anticlockwise).

Let there be required to write with the aid of an inequality the following arcs of the circle Σ (Fig. 41): (1) $\cup\Theta_0\Theta_1$; (2) $\cup\Theta_1\Theta_3$; (3) $\cup\Theta_1\Theta_0$; (4) $\cup\Theta_2\Theta_1$; (5) $\cup\Theta_0M$; (6) $\cup M\Theta_0$; (7) $\cup\Theta_3M$, where M is the midpoint of the arc $\Theta_1\Theta_2$.

(1) Point Θ_0 corresponds to the number 0, point Θ_1 to the number $\frac{\pi}{2}$, therefore the moving point of the arc $\Theta_0\Theta_1$ corresponds to a number x such that $0 \leqslant x \leqslant \frac{\pi}{2}$.

If a point of a circle corresponds to the number x, then it also corresponds to all numbers of the form $x + 2\pi k$ (k is an integer). We get therefore that the points of the arc $\Theta_0\Theta_1$ correspond to the numbers x satisfying the following system of inequalities:

$$0 + 2\pi k \leqslant x \leqslant \frac{\pi}{2} + 2\pi k \quad \text{or} \quad 2\pi k \leqslant x \leqslant \frac{\pi}{2} + 2\pi k.$$

This is an analytic notation of the arc $\Theta_0\Theta_1$.

(2) For the arc $\Theta_1\Theta_3$ we get: $\frac{\pi}{2} + 2\pi k \leqslant x \leqslant \frac{3\pi}{2} + 2\pi k$.

(3) As has been noted above, in this case the arc $\Theta_1\Theta_2\Theta_3\Theta_0$ is denoted by $\cup\Theta_1\Theta_0$. In the first tracing of the circle point Θ_1 corresponds to the number $\frac{\pi}{2}$, and point Θ_0 to the number 2π (but not to 0 since the circle is traced from Θ_1 to Θ_0 in the positive direction), hence, analytically, $\cup\Theta_1\Theta_0$ can be written in the following way:

$$\frac{\pi}{2} + 2\pi k \leqslant x \leqslant 2\pi + 2\pi k.$$

(4) The arc $\Theta_2\Theta_1$ may be written in two ways:

$$-\pi + 2\pi k \leqslant x \leqslant \frac{\pi}{2} + 2\pi k \text{ and } \pi + 2\pi n \leqslant x \leqslant \frac{5\pi}{2} + 2\pi n.$$

(5) $\cup\, \Theta_0 M$: $2\pi k \leqslant x \leqslant \frac{3\pi}{4} + 2\pi k$.

(6) $\cup\, M\Theta_0$: $\frac{3\pi}{4} + 2\pi k \leqslant x \leqslant 2\pi + 2\pi k$.

(7) $\cup\, \Theta_3 M$: $-\frac{\pi}{2} + 2\pi k \leqslant x \leqslant \frac{3\pi}{4} + 2\pi k$.

Remark. Presenting the arc in the form

$$\alpha + 2\pi k \leqslant x \leqslant \beta + 2\pi k, \qquad (1)$$

it is necessary to make sure that the inequality $\alpha < \beta$ is fulfilled, otherwise the system of inequalities (1) turns out to be contradictory.

Let now each quadrant of the circle be subdivided into three equal parts (Fig. 42). We are going to find analytic notations for the following arcs: (1) $\cup B_1B_2$; (2) $\cup \Theta_1 B_4$; (3) $\cup B_3A_1$; (4) $\cup A_2B_1$; (5) $\cup A_4\Theta_2$; (6) $\cup A_3B_2$.

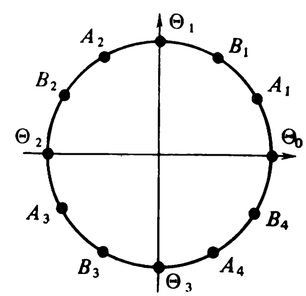

Fig. 42

(1) Consider the arc B_1B_2. Since each of the arcs Θ_0A_1, A_1B_1, $B_1\Theta_1$, Θ_1A_2, ..., $B_4\Theta_0$ has the length $\frac{\pi}{6}$, point B_1 corresponds to the number $\frac{\pi}{3}$, point B_2 to the number $\frac{5\pi}{6}$ when first tracing the circle counterclockwise. Consequently, the analytic notation of the arc B_1B_2 is

$$\frac{\pi}{3} + 2\pi k \leqslant x \leqslant \frac{5\pi}{6} + 2\pi k.$$

(2) $\cup\, \Theta_1 B_4$: $\frac{\pi}{2} + 2\pi k \leqslant x \leqslant \frac{11\pi}{6} + 2\pi k$.

(3) $\cup\, B_3A_1$: $-\frac{2\pi}{3} + 2\pi k \leqslant x \leqslant \frac{\pi}{6} + 2\pi k$ $\left(\text{or}\right]$ $\frac{4\pi}{3} + 2\pi n \leqslant x \leqslant \frac{13\pi}{6} + 2\pi n\Big)$

Remark. Here, we once again draw the reader's attention to the necessity to be careful with the notation of arc ends. Thus, when first tracing the circle, point B_3 corresponds to the number $\frac{4\pi}{3}$. When continuing motion from B_3 to A_1 and passing through point Θ_0, the point begins a second tracing of the circle, that is, point A_1

now corresponds to the number $\frac{13\pi}{6}$. Hence, we obtain the second notation for the arc B_3A_1.

(4) $\cup A_2B_1$: $-\frac{4\pi}{3}+2\pi k \leqslant x \leqslant \frac{\pi}{3}+2\pi k$ $\left(\text{or } \frac{2\pi}{3}+2\pi n \leqslant x \leqslant \frac{7\pi}{3}+2\pi n\right)$.

(5) $\cup A_4\Theta_2$: $-\frac{\pi}{3}+2\pi k \leqslant x \leqslant \pi+2\pi k$ $\left(\text{or } \frac{5\pi}{3}+2\pi n \leqslant x \leqslant 3\pi+2\pi n\right)$.

(6) $\cup A_3B_2$: $-\frac{5\pi}{6}+2\pi k \leqslant x \leqslant \frac{5\pi}{6}+2\pi k$ $\left(\text{or } \frac{7\pi}{6}+2\pi n \leqslant x \leqslant \frac{17\pi}{6}+2\pi n\right)$.

Example 1. Solve the inequality

$$\sin x > \frac{1}{2}. \tag{2}$$

Solution. By definition, $\sin x$ is the ordinate of point t of the circle Σ corresponding to the number x. We mark on the circle Σ the points with ordinates $\frac{1}{2}$ (points M and P in Fig. 43). The points whose ordinates exceed $\frac{1}{2}$ fill the open arc* MP. This arc is called the geometric solution of Inequality (2).

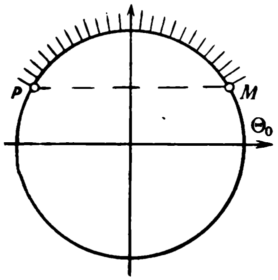

Fig. 43

Let us form the analytic notation of the open arc MP: $\frac{\pi}{6}+2\pi k < x < \frac{5\pi}{6}+2\pi k$. This is just the solution of Inequality (2).

Example 2. Solve the inequality

$$\cos x < \frac{1}{3}. \tag{3}$$

Solution. By definition, $\cos x$ is the abscissa of point $t \in \Sigma$, corresponding to the number x. We mark on the circle Σ the points with abscissas $\frac{1}{3}$ (points M and P in Fig. 44). Then the open arc MP is the geometric solution of Inequality (3) (the points belonging to this

* An arc with end-points deleted will be called an open arc.

arc have abscissas less than $\frac{1}{3}\big)$. Let us form the analytic notation of the open arc MP: $\arccos\frac{1}{3}+2\pi k<x<2\pi-\arccos\frac{1}{3}+2\pi k$.

Example 3. Solve the inequality

$$\tan x \leqslant -\frac{1}{2}. \tag{4}$$

Solution. Tan x is not defined for $x=\frac{\pi}{2}+\pi k$. To these numbers there correspond points Θ_1 and Θ_3 of the circle Σ (Fig. 45). Let us

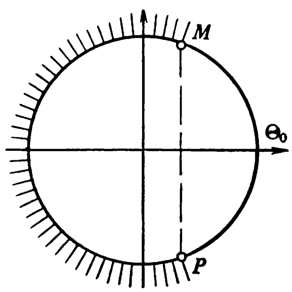

Fig. 44

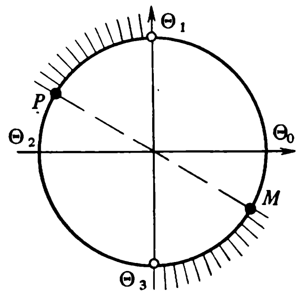

Fig. 45

mark on the semicircle $\Theta_3\Theta_1$ a point $M=M(x)$ such that $\tan x = -\frac{1}{2}$. Since on the arc $\Theta_3\Theta_1$ (or more precisely, on each of the intervals of the number line $\boldsymbol{R}$ mapped onto the arc $\Theta_3\Theta_1$) the function $y=\tan x$ increases, the inequality $\tan x \leqslant -\frac{1}{2}$ is fulfilled for all points of the arc $\Theta_3\Theta_1$ which are located clockwise from the point M, that is, on the half-open arc* Θ_3M.

Further, since the period of tangent is equal to π, Inequality (4) will also be fulfilled for all points of the arc Θ_1P which differs from the arc Θ_3M by π.

Thus, the geometric solution of Inequality (4) is represented by the union of two half-open arcs Θ_3M and Θ_1P. Let us form the analytic notations of the indicated arcs. For the arc Θ_3M we have: $-\frac{\pi}{2}+2\pi k<x\leqslant-\arctan\frac{1}{2}+2\pi k$, and for the arc Θ_1P: $\frac{\pi}{2}+2\pi k<x\leqslant\pi-\arctan\frac{1}{2}+2\pi k$.

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* An arc with only one end point deleted will be called a half-open arc.

However, the solution of Inequality (4) can be written more briefly: $-\frac{\pi}{2}+\pi n<x\leqslant-\arctan\frac{1}{2}+\pi n$.

Example 4. Solve the inequality

$$\cot x<\frac{\sqrt{3}}{3}. \tag{5}$$

Solution. Cot x is not defined for $x=\pi k$. To these numbers there correspond points Θ_0 and Θ_2 of the circle Σ (Fig. 46). Let us mark on the semicircle $\Theta_0\Theta_2$ a point $M=M(x)$ such that $\cot x=\frac{\sqrt{3}}{3}$. For this purpose, lay off the arc Θ_0M whose length is equal to $\operatorname{arccot}\frac{\sqrt{3}}{3}=\frac{\pi}{3}$. Since the function $y=\cot x$ decreases on the arc $\Theta_0\Theta_2$, the inequality $\cot x<\frac{\sqrt{3}}{3}$ is fulfilled for all points of the arc $\Theta_0\Theta_2$ which are located counterclockwise from point M, that is, on the open arc $M\Theta_2$ (when solving the inequality $\cot x>\frac{\sqrt{3}}{3}$, we would have to take the open arc Θ_0M). Taking into consideration that the period of cotangent is equal to π, let us also mark the arc $P\Theta_0$ on which Inequality (5) is fulfilled (it is obtained by rotating the arc $M\Theta_2$ about point O by 180°).

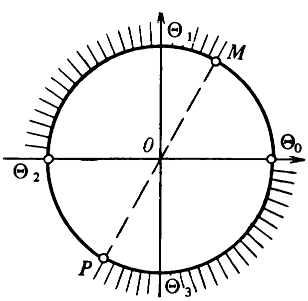

Fig. 46

Thus, the union of two open arcs $M\Theta_2$ and $P\Theta_0$ is the geometric solution of Inequality (5). The analytic notation of the arc $M\Theta_2$ is: $\frac{\pi}{3}+2\pi k<x<\pi+2\pi k$; the analytic notation of the arc $P\Theta_0$ is: $\frac{4\pi}{3}+2\pi k<x<2\pi+2\pi k$.

The solution of Inequality (5) can be written more briefly in the following way: $\frac{\pi}{3}+\pi k<x<\pi+\pi k$.

Example 5. Solve the system of inequalities

$$\begin{cases}\sin x<\frac{\sqrt{3}}{2}\\ \cos x>-\frac{\sqrt{2}}{2}.\end{cases} \tag{6}$$

Solution. Let us find a geometric solution of the inequality $\sin x<\frac{\sqrt{3}}{2}$ (the arc MP of the circle Σ is indicated in Fig. 47 by internal

hatching). On the same circle, we find the geometric solution of the inequality $\cos x > -\frac{\sqrt{2}}{2}$ (the corresponding arc EK is shown in Fig. 47 by external hatching). Then, the intersection of the arcs MP and EK, that is, the union of the arcs MK and EP, is the geometric solution of System (6). It only remains to form the analytic notation of either of these arcs. For the arc MK we have: $\frac{2\pi}{3}+2\pi k < x < \frac{3\pi}{4}+2\pi k$; for the arc EP we have: $-\frac{3\pi}{4}+2\pi k < x < \frac{\pi}{3}+2\pi k$.

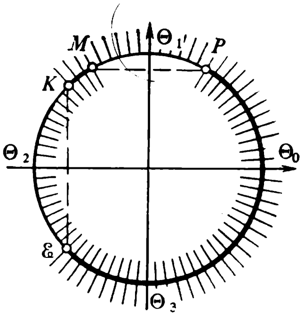

Fig. 47

Example 6. Solve the inequality

$$2\sin^2\left(x+\frac{\pi}{4}\right)+\sqrt{3}\cos 2x > 0. \quad (7)$$

Solution. Applying the formula $1-\cos 2\alpha = 2\sin^2\alpha$, we transform Inequality (7) to

$$1-\cos\left(2x+\frac{\pi}{2}\right)+\sqrt{3}\cos 2x > 0,$$

and further

$$-\cos\left(2x+\frac{\pi}{2}\right)+\sqrt{3}\cos 2x > -1, \ \sin 2x+\sqrt{3}\cos 2x > -1,$$

$$\frac{1}{2}\sin 2x+\frac{\sqrt{3}}{2}\cos 2x > -\frac{1}{2}, \ \sin\frac{\pi}{6}\sin 2x+\cos\frac{\pi}{6}\cos 2x > -\frac{1}{2},$$

$$\cos\left(2x-\frac{\pi}{6}\right) > -\frac{1}{2}. \quad (8)$$

We then solve Inequality (8). Setting $t=\left(2x-\frac{\pi}{6}\right)$, we get the inequality: $\cos t > -\frac{1}{2}$ whose solution is found with the aid of the circle (Fig. 48): $-\frac{2\pi}{3}+2\pi k < t < \frac{2\pi}{3}+2\pi k$. Returning to the variable x, we get: $-\frac{2\pi}{3}+2\pi k < 2x-\frac{\pi}{6} < \frac{2\pi}{3}+2\pi k$, whence $-\frac{\pi}{4}+\pi k < x < \frac{5\pi}{12}+\pi k$ which is the solution of Inequality (8), that is, also of Inequality (7).

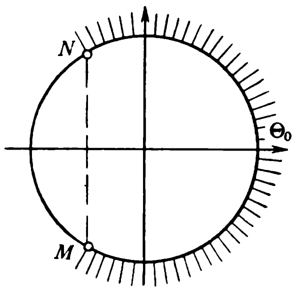

Fig. 48

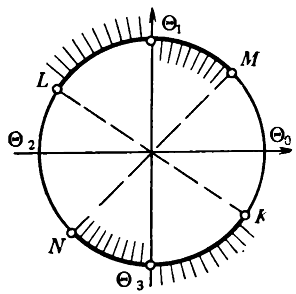

Fig. 49

Example 7. Solve the inequality

$$6 \sin^2 x - \sin x \cos x - \cos^2 x > 2. \quad (9)$$

Solution. Since $2 = 2(\sin^2 x + \cos^2 x)$, we transform Inequality (9) to

$$4 \sin^2 x - \sin x \cos x - 3 \cos^2 x > 0. \quad (10)$$

Since $\cos^2 x \geqslant 0$, Inequality (10) is equivalent to the following collection of systems:

$$\begin{cases} \cos^2 x = 0 \\ 4 \sin^2 x > 0 \end{cases}; \quad \begin{cases} \cos^2 x > 0 \\ 4 \tan^2 x - \tan x - 3 > 0. \end{cases} \quad (11)$$

The first system of Collection (11) has the following solution: $x = \frac{\pi}{2} + \pi k$. The second system of this collection is equivalent to the following system:

$$\begin{cases} x \neq \frac{\pi}{2} + \pi k \\ (\tan x - 1)\left(\tan x + \frac{3}{4}\right) > 0, \end{cases}$$

which, in turn, is equivalent to the collection of inequalities:

$$\tan x < -\frac{3}{4}; \quad \tan x > 1. \quad (12)$$

Let us find the solution of Collection (12). The union of the open arcs $M\Theta_1$ and $N\Theta_3$ (shown in Fig. 49 by internal hatching) is the geometric solution of the inequality $\tan x > 1$, while the union of the open arcs $\Theta_1 L$ and $\Theta_3 K$ (indicated by external hatching) is the geometric solution of the inequality $\tan x < -\frac{3}{4}$. The geometric solution of Collection (12) is represented by the union of four arcs:

$M\Theta_1$, $N\Theta_3$, $\Theta_1 L$, $\Theta_3 K$. Further, since the geometric solution of the first system of Collection (11) represents a two-element set $\{\Theta_1, \Theta_3\}$, the geometric solution of this collection is represented by the union of two arcs: ML and NK. Let us form the analytic notation of the arc ML: $\frac{\pi}{4} + 2\pi k < x < \pi - \arctan\frac{3}{4} + 2\pi k$.

Taking into consideration that the arc NK is obtained by rotating the arc ML about point O by 180°, we may avoid the analytic notation of the arc NK and write at once the solution of the collection of Systems (12) in the form: $\frac{\pi}{4} + \pi k < x < \pi - \arctan\frac{3}{4} + \pi k$.

This is just the solution of Inequality (9).

Example 8. Solve the inequality

$$\sin x + \cos x < \frac{1}{\sin x}. \tag{13}$$

Solution. We have in succession:

$$\sin x + \cos x - \frac{1}{\sin x} < 0, \quad \frac{\sin^2 x + \sin x \cos x - 1}{\sin x} < 0,$$
$$\frac{\sin x \cos x - \cos^2 x}{\sin x} < 0, \quad \frac{\cos x (\sin x - \cos x)}{\sin x} < 0. \tag{14}$$

Let us use the identity $\sin x = \tan x \cos x$. Here we reduce the domain of definition of the inequality, but do not lose solutions since the values of x for which $\cos x = 0$ are not solutions of Inequality (14).

Inequality (14) is transformed to the form:

$$\frac{\cos^2 x (\tan x - 1)}{\sin x} < 0, \text{ and further } \frac{\tan x - 1}{\sin x} < 0. \tag{15}$$

The obtained inequality is equivalent to the following collection of systems of inequalities:

$$\left[\begin{array}{l} \begin{cases} \tan x > 1 & (16) \\ \sin x < 0, \end{cases} \\ \begin{cases} \tan x < 1 \\ \sin x > 0. & (17) \end{cases} \end{array}\right.$$

Let us solve System (16). Figure 50 shows the union of the arcs $P\Theta_3$ and $M\Theta_1$ which represents a geometric solution of the inequality $\tan x > 1$, and the arc $\Theta_2\Theta_0$ which gives a geometric solution of the inequality $\sin x < 0$. The geometric solution of System (16) is

represented by the arc $P\Theta_3$, the analytic notation having the form: $\frac{5\pi}{4}+2\pi k<x<\frac{3\pi}{2}+2\pi k$. This is just the solution of System (16).

Let us solve System (17). As is seen from Fig. 51, the geometric solution of this system is the union of arcs $\Theta_0 M$ and $\Theta_1\Theta_2$. The

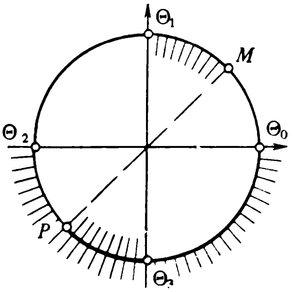

Fig. 50

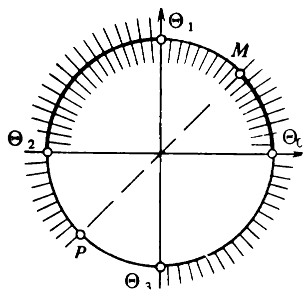

Fig. 51

analytic notation has the form:

$$2\pi k<x<\frac{\pi}{4}+2\pi k;\ \frac{\pi}{2}+2\pi k<x<\pi+2\pi k.$$

Hence, the solution of the collection of Systems (16) and (17), and, at the same time, the solution of Inequality (13) is as follows: $\frac{5\pi}{4}+2\pi k<x<\frac{3\pi}{2}+2\pi k$; $2\pi k<x<\frac{\pi}{4}+2\pi k$; $\frac{\pi}{2}+2\pi k<x<\pi+2\pi k$.

Let us note in conclusion that it is not always possible to solve a system or a collection of trigonometric inequalities with the aid of the circle. This is the case, for instance, when the least common multiple of the periods of all the functions entering the inequalities forming a given system or collection is greater than the circumference of the circle, that is, greater than 2π. In such cases, instead of the circle, the number line is used.

EXERCISES

In Problems 1604 and 1605 solve the given simplest inequalities:

1604. (1) $\sin x>-\frac{1}{2}$; (2) $\cos x<\frac{\sqrt{3}}{2}$; (3) $\tan x\geqslant-\frac{\sqrt{3}}{3}$; (4) $\cot x\leqslant-1$.

1605. (1) $\sin x<\frac{1}{5}$; (2) $\cos x\geqslant-0.7$; (3) $\tan x\leqslant 5$; (4) $\cot x>-\frac{\sqrt{3}}{4}$.

In Problems 1606 through 1613, solve the indicated inequalities:

1606. $\begin{cases} \sin x < \frac{1}{2} \\ \cos x < \frac{1}{2}. \end{cases}$ 1607. $\begin{cases} \sin x > -\frac{\sqrt{3}}{2} \\ \tan x \leqslant 0. \end{cases}$

1608. $\begin{cases} \cos x \leqslant \frac{\sqrt{2}}{2} \\ \cot x > -\sqrt{3}. \end{cases}$ 1609. $\begin{cases} \tan x < 1 \\ \cot x \geqslant -\frac{\sqrt{3}}{3}. \end{cases}$

1610. $\begin{cases} \sin x > \frac{1}{5} \\ \cos x < \frac{1}{5}. \end{cases}$ 1611. $\begin{cases} \cos x \geqslant -\frac{3}{5} \\ \tan x < 3. \end{cases}$

1612. $\begin{cases} \sin x < \frac{4}{7} \\ \cos x < 2. \end{cases}$ 1613. $\begin{cases} \tan x > 0.23 \\ \cot x \leqslant 0.3. \end{cases}$

In Problems 1614 through 1617, solve the given collections of inequalities:

1614. $\sin x > \frac{2}{3}$; $\cos x < 0$. 1615. $\cos x < \frac{1}{2}$; $\tan x > -3.5$.

1616. $\sin x < -\frac{\sqrt{3}}{2}$; $\cot x \leqslant 7$. 1617. $\tan x < \frac{\sqrt{3}}{3}$; $\cot x < \sqrt{2}$.

In Problems 1618 through 1651, solve the given inequalities:

1618. $\cos x^2 \geqslant \frac{1}{2}$. 1619. $\sqrt{3} \sin 2x + \cos 2x < 1$.

1620. $\cos 3x + \sqrt{3} \sin 3x < -\sqrt{2}$. 1621. $\cos 2x + \cos x > 0$.

1622. $\frac{\cos x}{1 + \cos 2x} < 0$. 1623. $\sin 3x > \cos 3x$.

1624. $\tan x + 3 \cot x - 4 > 0$. 1625. $\sin^2 x - \cos^2 x - 3 \sin x + 2 < 0$.

1626. $2 \sin^2 \frac{x}{2} + \cos 2x < 0$.

1627. $\tan^3 x + 3 > 3 \tan x + \tan^2 x$. 1628. $\frac{\sin 3x - \cos 3x}{\sin 3x + \cos 3x} < 0$.

1629. $5 \sin^2 x - 3 \sin x \cos x - 36 \cos^2 x > 0$.

1630. $2 \sin^2 x - 4 \sin x \cos x + 9 \cos^2 x > 0$.

1631. $\cos^2 x + 3 \sin^2 x + 2\sqrt{3} \sin x \cos x < 1$.

1632. $3 \sin^2 x + \sin 2x - \cos^2 x \geqslant 2$. 1633. $\sqrt{3} \cos^{-2} x < 4 \tan x$.

1634. $\sin 4x + \cos 4x \cot 2x > 1$. 1635. $2 + \tan 2x + \cot 2x < 0$.

1636. $2 \cos x (\cos x - \sqrt{8} \tan x) < 5$. 1637. $\sin x + \cos x < \frac{1}{\cos x}$.

1638. $\sin^6 x + \cos^6 x < \frac{7}{16}$. 1639. $\cot x + \frac{\sin x}{\cos x - 2} \geqslant 0$.

1640. $\cos^2 2x + \cos^2 x \leqslant 1$. 1641. $8 \sin^2 \frac{x}{2} + 3 \sin x - 4 > 0$.

1642. $\sin x + \cos x > \sqrt{2} \cos 2x$. 1643. $\tan x + \tan 2x + \tan 3x > 0$.

1644. $\cos 2x \cos 5x < \cos 3x$. 1645. $\sin 2x \sin 3x - \cos 2x \cos 3x > \sin 10x$.

1646. $\cot x + \cot \left(x + \frac{\pi}{2}\right) + 2 \cot \left(x + \frac{\pi}{3}\right) > 0$.

1647. $2 \sin^2 x - \sin x + \sin 3x < 1$. 1648. $4 \sin x \sin 2x \sin 3x > \sin 4x$.

1649. $\frac{\cos^2 2x}{\cos^2 x} \geqslant 3 \tan x$. 1650. $3 \cos^2 x \sin x - \sin^2 x < \frac{1}{2}$.

1651. $\frac{\cos x + 2 \cos^2 x + \cos 3x}{\cos x + 2 \cos^2 x - 1} > 1$.

In Problems 1652 and 1653, solve the indicated systems of inequalities:

1652. $\begin{cases} \cos x < 0 \\ \sin \frac{3}{5} x > 0. \end{cases}$ 1653. $\begin{cases} \sin \frac{x}{2} < \frac{1}{2} \\ \cos 2x > -\frac{1}{2}. \end{cases}$

SEC. 27. PARAMETRIC EQUATIONS AND INEQUALITIES

Example 1. Solve the equation

$$\sin^4 x + \cos^4 x = a, \tag{1}$$

Solution. Applying the power reduction formulas, we get: $\left(\frac{1 - \cos 2x}{2}\right)^2 + \left(\frac{1 + \cos 2x}{2}\right)^2 = a$, and further

$$\cos^2 2x = 2a - 1. \tag{2}$$

Let us find the singular values of the parameter (see Sec. 20), that is, the values for which the right-hand side of the equation is equal to either 0 or 1 (if $2a - 1 < 0$ or $2a - 1 > 1$, then the equation has no solution).

If $2a - 1 = 0$, then $a = \frac{1}{2}$; if $2a - 1 = 1$, then $a = 1$.

Thus, we shall consider Equation (2) in each of the following five cases: (1) $a < \frac{1}{2}$; (2) $a = \frac{1}{2}$; (3) $\frac{1}{2} < a < 1$; (4) $a = 1$; (5) $a > 1$.

(1) If $a < \frac{1}{2}$, then $2a - 1 < 0$, and Equation (2) has no root.

(2) If $a = \frac{1}{2}$, then Equation (2) takes the form $\cos^2 2x = 0$, whence we find: $x = \frac{\pi}{4} + \frac{\pi}{2} k$.

(3) If $\frac{1}{2} < a < 1$, then $0 < 2a - 1 < 1$. Let us transform Equation (2) to $\frac{1 + \cos 4x}{2} = 2a - 1$, and further $\cos 4x = 4a - 3$. Since in the case under consideration $\frac{1}{2} < a < 1$, we have: $2 < 4a < 4$, and then $-1 < 4a - 3 < 1$. Hence, the equation $\cos 4x = 4a - 3$ has the solution $4x = \pm \arccos (4a - 3) + 2\pi k$, whence

$$x = \pm \frac{1}{4} \arccos (4a - 3) + \frac{\pi}{2} k. \qquad (3)$$

(4) If $a = 1$, then Equation (2) takes the form $\cos^2 2x = 1$. From this equation we find: $x = \frac{\pi}{2} k$.

(5) If $a > 1$, then $2a - 1 > 1$ and Equation (2) has no root.

Note that if $a = \frac{1}{2}$ or $a = 1$, then the solution may also be written in the form (3).

Answer: (1) if $a < \frac{1}{2}$; $a > 1$, then there is no root;

(2) if $\frac{1}{2} \leqslant a \leqslant 1$, then $x = \pm \arccos \frac{1}{4} (4a - 3) + \frac{\pi}{2} k$.

Example 2. Solve the equation

$$(a - 1) \sin^2 x - 2 (a + 1) \sin x + 2a - 1 = 0. \qquad (4)$$

Solution. Let us set $y = \sin x$, then Equation (4) will take the form:

$$(a - 1) y^2 - 2 (a + 1) y + 2a - 1 = 0. \qquad (5)$$

The first singular value of the parameter a is the value $a = 1$ which makes the coefficient of y^2 vanish.

For $a = 1$ Equation (5) takes the form: $-4y + 1 = 0$, whence we find: $y = \frac{1}{4}$, i.e. $\sin x = \frac{1}{4}$, and, consequently,

$$x = (-1)^k \arcsin \frac{1}{4} + \pi k.$$

Consider now the case when $a \neq 1$. Let us find the discriminant of Equation (5). We have: $\frac{D}{4} = (a+1)^2 - (a-1)(2a-1) = -a^2 + 5a$.

The second singular value of the parameter a is represented by the values for which $D=0$. These values are: $a=0$, $a=5$. Note that $D<0$ if $a<0$ or $a>5$, and $D \geqslant 0$ if $0 \leqslant a \leqslant 5$.

Hence, we have to consider Equation (5) in each of the following cases: $a<0$, $\begin{cases} 0 \leqslant a \leqslant 5 \\ a \neq 1 \end{cases}$; $a>5$.

If $a<0$ or $a>5$, Equation (5) has no root.

In the case $\begin{cases} 0 \leqslant a \leqslant 5 \\ a \neq 1 \end{cases}$ the quadratic equation (5) has two real roots:

$$y_{1,2} = \frac{a+1 \pm \sqrt{5a-a^2}}{a-1}.$$

Since $y = \sin x$, the following inequalities must be fulfilled: $-1 \leqslant y_1 \leqslant 1$, $-1 \leqslant y_2 \leqslant 1$.

It is easy to note that the value $y_1 = \frac{a+1+\sqrt{5a-a^2}}{a-1}$ satisfies the double inequality $-1 \leqslant y_1 \leqslant 1$ only for $a=0$. Indeed, if $a=0$, then $y=-1$, if $a>0$, then $|a+1| > |a-1|$ and the more so $|a+1+\sqrt{5a-a^2}| > |a-1|$, i.e. $|y_1| > 1$.

If $a=0$, then the equation $\sin x = y_1$ takes the form: $\sin x = -1$, whence we find: $x = -\frac{\pi}{2} + 2\pi k$.

Let us now look for the values of the parameter a from the set under consideration: $\begin{cases} 0 \leqslant a \leqslant 5 \\ a \neq 1 \end{cases}$ which satisfy the system of inequalities $-1 \leqslant y_2 \leqslant 1$, that is, the system

$$\begin{cases} \dfrac{a+1-\sqrt{5a-a^2}}{a-1} \geqslant -1 \\ \dfrac{a+1-\sqrt{5a-a^2}}{a-1} \leqslant 1. \end{cases} \qquad (6)$$

System (6) is, in turn, equivalent to the following collection of systems of inequalities:

$$\begin{cases} a-1>0 \\ a+1-\sqrt{5a-a^2} \geqslant 1-a; \\ a+1-\sqrt{5a-a^2} \leqslant a-1 \end{cases} \quad \begin{cases} a-1<0 \\ a+1-\sqrt{5a-a^2} \leqslant 1-a \\ a+1-\sqrt{5a-a^2} \geqslant a-1. \end{cases} \tag{7}$$

Let us solve the first system of Collection (7). We have:

$\begin{cases} a>1 \\ \sqrt{5a-a^2} \leqslant 2a \\ \sqrt{5a-a^2} \geqslant 2, \end{cases}$ and further $\begin{cases} a>1 \\ 5a-a^2 \leqslant 4a^2 \\ 5a-a^2 \geqslant 4, \end{cases}$ whence we find: $1<a\leqslant 4$.

Solving the second system of Collection (7), we have:

$\begin{cases} a<1 \\ \sqrt{5a-a^2} \geqslant 2a \\ \sqrt{5a-a^2} \leqslant 2, \end{cases}$ and further (since $a\geqslant 0$) $\begin{cases} a<1 \\ 5a-a^2 \geqslant 4a^2 \\ 5a-a^2 \leqslant 4, \end{cases}$ whence we find: $0\leqslant a<1$.

Thus, the collection of Systems (7), and, consequently, System (6) have the following solutions: $0\leqslant a<1$; $1<a\leqslant 4$. This means that on the set $\begin{cases} 0\leqslant a\leqslant 5 \\ a\neq 1 \end{cases}$ the equation

$$\sin x = \frac{a+1-\sqrt{5a-a^2}}{a-1} \tag{8}$$

has a solution only if $\begin{cases} 0\leqslant a\leqslant 4 \\ a\neq 1. \end{cases}$ This solution is: $x=(-1)^k \times \arcsin \dfrac{a+1-\sqrt{5a-a^2}}{a-1}+\pi k$. Note that this notation also includes the case considered above if $a=0$.

If $4<a\leqslant 5$, then Equation (8), and, hence, Equation (4) have no root.

Answer: (1) if $a=1$, then $x=(-1)^k \arcsin \frac{1}{4}+\pi k$;

(2) if $\begin{cases} 0\leqslant a\leqslant 4 \\ a\neq 1, \end{cases}$ then $x=(-1)^k \arcsin \dfrac{a+1-\sqrt{5a-a^2}}{a-1}+\pi k$;

(3) if $a<0$; $a>4$, then Equation (4) has no root.

Example 3. Solve the equation

$$\cos(a+x)=\frac{\cos a}{\cos x}. \tag{9}$$

Solution. Multiplying both sides of Equation (9) by $\cos x$, we get: $\cos x\cos(a+x)=\cos a$, and further

$$\cos(x+a+x)+\cos(x-a-x)=2\cos a,$$

that is,

$$\cos(2x+a)=\cos a. \tag{10}$$

From Equation (10) we find:

$$x=\pi k;\quad x=-a+\pi k. \tag{11}$$

Check. In the process of solving Equation (9) we multiplied both sides of the equation by $\cos x$ which led to an extension of the domain of definition of the equation, and, hence, might also cause the appearance of extraneous roots. Let us choose from the found collection of families of solutions of Equation (10) the families which are solutions of Equation (9). To this end, we eliminate from Collection (11) the values of x for which $\cos x=0$, that is, the values $x=\frac{\pi}{2}+\pi n$. It is clear that the families $x=\pi k$ and $x=\frac{\pi}{2}+\pi n$ do not intersect. Then, setting $-a+\pi k=\frac{\pi}{2}+\pi n$, we find: $a=\frac{\pi}{2}(-1-2k+2n)$. This means that the family $x=a+\pi k$ is a solution of Equation (9) only for the values $a\neq\frac{\pi}{2}(2n-2k-1)$, or, more briefly, for $a\neq\frac{\pi}{2}(2l-1)$, where $l=n-k$ $(l=0;\ \pm1;\ \pm2;\ \ldots)$.

Answer: (1) if $a=\frac{\pi}{2}(2l-1)$, then $x=\pi k$;

(2) if $a\neq\frac{\pi}{2}(2l-1)$, then $x=\pi k$; $x=-a+\pi k$.

Example 4. Solve the system of equations

$$\begin{cases}\sin x\cos y=a^2\\ \sin y\cos x=a.\end{cases} \tag{12}$$

Solution. Replacing the first equation of System (12) by the sum of the first and second equations and the second equation by the

difference between them, we get the system equivalent to System (12)

$$\begin{cases} \sin x \cos y + \sin y \cos x = a^2 + a \\ \sin x \cos y - \sin y \cos x = a^2 - a \end{cases}$$

or $\qquad\qquad\qquad\qquad\qquad\qquad$ (13)

$$\begin{cases} \sin (x + y) = a^2 + a \\ \sin (x - y) = a^2 - a. \end{cases}$$

It is clear that System (13) has solutions if and only if the parameter a satisfies the following system of inequalities:

$$\begin{cases} |a^2 + a| \leqslant 1 \\ |a^2 - a| \leqslant 1. \end{cases} \qquad (14)$$

System (14) is equivalent to the following system:

$$\begin{cases} a^2 + a \leqslant 1 \\ a^2 + a \geqslant -1 \\ a^2 - a \leqslant 1 \\ a^2 - a \geqslant -1 \end{cases} \quad \text{or} \quad \begin{cases} a^2 + a - 1 \leqslant 0 \\ a^2 + a + 1 \geqslant 0 \\ a^2 - a - 1 \leqslant 0 \\ a^2 - a + 1 \geqslant 0. \end{cases} \qquad (15)$$

The second and fourth inequalities of System (15) are fulfilled for any a since the quadratic trinomials in their left-hand sides have negative discriminants and positive leading coefficients. Hence, System (15) is equivalent to the following system: $\begin{cases} a^2 + a - 1 \leqslant 0 \\ a^2 - a - 1 \leqslant 0 \end{cases}$. Solving this system, we find:

$$-\frac{\sqrt{5}-1}{2} \leqslant a \leqslant \frac{\sqrt{5}-1}{2}.$$

System (13) has a solution only for these values of the parameter a.

Thus, let $-\frac{\sqrt{5}-1}{2} \leqslant a \leqslant \frac{\sqrt{5}-1}{2}$. From System (13) we get:

$$\begin{cases} x + y = (-1)^k \arcsin (a^2 + a) + \pi k, \\ x - y = (-1)^n \arcsin (a^2 - a) + \pi n, \end{cases}$$

and further:

$$\begin{cases} x = \frac{1}{2} ((-1)^k \arcsin (a^2 + a) + (-1)^n \arcsin (a^2 - a) + \pi k + \pi n), \\ y = \frac{1}{2} ((-1)^k \arcsin (a^2 + a) - (-1)^n \arcsin (a^2 - a) + \pi k - \pi n). \end{cases}$$

Answer:

(1) if $a < -\frac{\sqrt{5}-1}{2}$; $a > \frac{\sqrt{5}-1}{2}$, then there is no solution;

(2) if $-\frac{\sqrt{5}-1}{2} \leqslant a \leqslant \frac{\sqrt{5}-1}{2}$, then

$$\begin{cases} x = \frac{\alpha+\beta+\pi(k+n)}{2} \\ y = \frac{\alpha-\beta+\pi(k-n)}{2}, \end{cases}$$

where $\alpha = (-1)^k \arcsin(a^2+a)$, $\beta = (-1)^n \arcsin(a^2-a)$

Example 5. Solve the inequality

$$\tan x + \cot x \leqslant a. \tag{16}$$

Solution. We transform Inequality (16) to

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \leqslant a, \text{ and further } \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} \leqslant a,$$

that is,

$$\frac{2}{\sin 2x} \leqslant a. \tag{17}$$

Let us set $y = \sin 2x$. Then Inequality (17) will take the form $\frac{2}{y} \leqslant a$, and the problem will be reduced to solving the following system of inequalities:

$$\begin{cases} \frac{2}{y} \leqslant a \\ -1 \leqslant y \leqslant 1, \end{cases} \text{ i.e. } \begin{cases} \frac{ay-2}{y} \geqslant 0 \\ -1 \leqslant y \leqslant 1. \end{cases} \tag{18}$$

Note that $a = 0$ is a singular value of the parameter a. Hence, we have to consider three cases: (1) $a = 0$; (2) $a > 0$; (3) $a < 0$.

(1) If $a = 0$, then System (18) takes the form $\begin{cases} -\frac{2}{y} \geqslant 0 \\ -1 \leqslant y \leqslant 1, \end{cases}$ whence we find $-1 \leqslant y < 0$.

(2) If $a > 0$, then System (18) is transformed to $\begin{cases} \frac{y - \frac{2}{a}}{y} \geqslant 0 \\ -1 \leqslant y \leqslant 1, \end{cases}$

whence we find:

$$\begin{cases} y < 0, \ y \geqslant \frac{2}{a} \\ -1 \leqslant y \leqslant 1. \end{cases} \tag{19}$$

Here, the value $a = 2$ is a singular value of the parameter, therefore we have to consider three cases: (a) $0 < a < 2$; (b) $a = 2$; (c) $a > 2$.

(a) If $0<a<2$, then $\frac{2}{a} > 1$, and System (19) has the following solution: $-1 \leqslant y < 0$;

(b) if $a = 2$, then System (19) has the following solution: $-1 \leqslant y < 0$; $y = 1$;

(c) if $a > 2$, then System (19) has the following solution: $-1 \leqslant y < 0$; $\frac{2}{a} \leqslant y \leqslant 1$.

(3) If $a < 0$, then System (18) is transformed to $\begin{cases} \dfrac{y-\frac{2}{a}}{y} \leqslant 0 \\ -1 \leqslant y \leqslant 1, \end{cases}$ and further,

$$\begin{cases} \frac{2}{a} \leqslant y < 0 \\ -1 \leqslant y < 1. \end{cases} \tag{20}$$

Here, the singular value of the parameter is represented by the value $a = -2$. Therefore we have to consider the following three cases: (a) $a < -2$; (b) $a = -2$; (c) $-2 < a < 0$.

(a) If $a < -2$, we have $\frac{2}{a} > -1$, and from System (20) we find $\frac{2}{a} \leqslant y < 0$.

(b) If $a = -2$, then from System (20) we find $-1 \leqslant y < 0$.

(c) Finally if $-2 < a < 0$, we have $\frac{2}{a} < -1$, and System (20) has the following solution: $-1 \leqslant y < 0$.

Summing up, we get the following solution of System (18):

(1) if $a < -2$, then $\frac{2}{a} \leqslant y < 0$;

(2) if $-2 \leqslant a < 2$, then $-1 \leqslant y < 0$;

(3) if $a = 2$, then $-1 \leqslant y < 0$; $y = 1$;

(4) if $a > 2$, then $-1 \leqslant y < 0$; $\frac{2}{a} \leqslant y \leqslant 1$.

Since $y = \sin 2x$, we get:

1. If $a < -2$, then $\frac{2}{a} \leqslant \sin 2x < 0$, whence (Fig. 52)

$$2\pi k + \pi < 2x \leqslant \pi + \arcsin\left(-\frac{2}{a}\right) + 2\pi k;$$

$$2\pi k + \arcsin\frac{2}{a} \leqslant 2x < 2\pi k,$$

and hence,

$$\pi k + \frac{\pi}{2} < x \leqslant \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{2}{a} + \pi k;\ \pi k + \frac{1}{2}\arcsin\frac{2}{a} \leqslant x < \pi k.$$

2. If $-2 \leqslant a < 2$, then $-1 \leqslant \sin 2x < 0$, whence

$$2\pi k - \pi < 2x < 2\pi k,$$

and hence $\pi k - \frac{\pi}{2} < x < \pi k$.

3. If $a = 2$, then from the system of inequalities $-1 \leqslant \sin 2x < 0$ we get $\pi k - \frac{\pi}{2} < x < \pi k$, and from the equation $\sin 2x = 1$ we

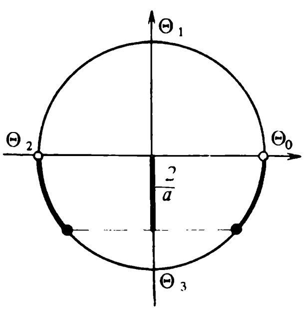

Fig. 52

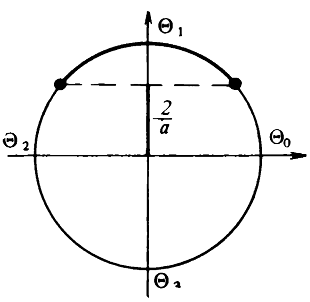

Fig. 53

find:

$$x = \frac{\pi}{4} + \pi k.$$

4. If $a > 2$, then from the system of inequalities $-1 \leqslant \sin 2x < 0$ we find (as above) $\pi k - \frac{\pi}{2} < x < \pi k$, and from the system $\frac{2}{a} \leqslant \sin 2x \leqslant 1$ we have (Fig. 53):

$$2\pi k + \arcsin\frac{2}{a} \leqslant 2x \leqslant \pi - \arcsin\frac{2}{a} + 2\pi k,$$

whence

$$\pi k + \frac{1}{2}\arcsin\frac{2}{a} \leqslant x \leqslant \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{2}{a} + \pi k.$$

Answer: (1) if $a < -2$, then $\pi k + \frac{\pi}{2} < x \leqslant \frac{\pi}{2} - \alpha + \pi k$; $\pi k + \alpha \leqslant x < \pi k$;

(2) if $-2 \leqslant a < 2$, then $\pi k - \frac{\pi}{2} < x < \pi k$;

(3) if $a = 2$, then $\pi k - \frac{\pi}{2} < x < \pi k$; $x = \frac{\pi}{4} + \pi k$;

(4) if $a > 2$, then $\pi k - \frac{\pi}{2} < x < \pi k$; $\pi k + \alpha \leqslant x \leqslant \frac{\pi}{2} - \alpha + \pi k$, where $\alpha = \frac{1}{2} \arcsin \frac{2}{a}$.

EXERCISES

In Problems 1654 through 1686, solve the given equations:

1654. $\cos 2x - \cos 4x = a \sin x$. **1655.** $12 \sin x + 4\sqrt{3} \cos(\pi + x) = a\sqrt{3}$.

1656. $\sin(x - a) = \sin x + \sin a$. **1657.** $\sin(a + x) + \sin x = \cos \frac{a}{2}$.

1658. $a(\sin x + \cos x)^2 = b \cos 2x$. **1659.** $(a - 1)\cos x + (a + 1)\sin x = 2a$.

1660. $\sin(x + a) + \cos(x + a) = \sin(x - a) + \cos(x - a)$.

1661. $1 + \sin^2 ax = \cos x$. **1662.** $\sin^6 x + \cos^6 x = a$.

1663. $\sin^4 x + \cos^4 x + \sin 2x + a = 0$. **1664.** $\tan x + \tan a + 1 = \tan x \tan a$.

1665. $a \cos^2 \frac{x}{2} - (a + 2b)\sin^2 \frac{x}{2} = a \cos x - b \sin x$. **1666.** $\frac{\tan ax}{\sin bx} = 0$.

1667. $\sin 3x = a \sin x$. **1668.** $\cos 3x = a \cos x$. **1669.** $2\cos(a + x) = \frac{\cos a}{\cos x}$.

1670. $\sin(x + a) = \frac{\cos a}{\sin x}$. **1671.** $\cos x - \sin a + 2 \cos 3x \sin(a - 3x) = 0$.

1672. $a^2 - 2a + \frac{1}{\cos^2 \pi (a + x)} = 0$. **1673.** $\sin x + 2 \cos ax = 3$.

1674. $\sin^2 x + 4 \sin x + a = 0$. **1675.** $\cos^2 x - 3 \cos x + a = 0$.

1676. $\sin^4 x - 2\cos^2 x + a^2 = 0$. **1677.** $\frac{a + \sin x}{a \cos x + 1} = \frac{a + \cos x}{a \sin x + 1}$.

1678. $\tan^2 x + \tan(a + x)\tan(a - x) = 0$. **1679.** $\tan^2 x - 2 \tan a \tan x + 1 = 0$.

1680. $\sin x \tan x + 2 \cos x = a$. **1681.** $\sin a \tan^2 x - 2 \cos a \tan x + 1 = 0$.

1682. $\arctan a - \arctan \frac{a - 1}{a + 1} = \arctan x$.

1683. $\arctan \frac{1}{x - 1} - \arctan \frac{1}{x + 1} = \arctan a$.

1684. $\sin 3x + \sin 2x = a \sin x$. **1685.** $(\sin x + \cos x)\sin 2x = a(\sin^3 x + \cos^3 x)$.

1686. $\sin^2 x - \sin x \cos x - 2\cos^2 x = a$.

In Problems 1687 through 1697, solve the indicated systems of equations:

1687. $\begin{cases} \sin x + \sin y = a \\ x + y = b. \end{cases}$ **1688.** $\begin{cases} \cos x - \cos y = a \\ x + y = b. \end{cases}$

1689. $\begin{cases} \sin x \sin y = a \\ x + y = b. \end{cases}$ **1690.** $\begin{cases} \sin x \cos y = a \\ x + y = b. \end{cases}$

1691. $\begin{cases} \sin^2 x - \sin^2 y = a \\ x + y = b. \end{cases}$ **1692.** $\begin{cases} \sin x \sin y = a \\ \cos x \cos y = 3a. \end{cases}$

1693. $\begin{cases} \sin x \cos y = 2a \\ \cos x \sin y = a. \end{cases}$ **1694.** $\begin{cases} \cot x + \cot y = a \\ x + y = b. \end{cases}$

1695. $\begin{cases} \sin x \cos 2y = a^2 + 1 \\ \cos x \sin 2y = a. \end{cases}$ **1696.** $\begin{cases} x - y = a \\ 2(\cos 2x + \cos 2y) = 1 + 4\cos^2(x - y). \end{cases}$

1697. $\begin{cases} \sin x + \sin y = a \\ \sin x \sin y = -2a^2. \end{cases}$

In Problems 1698 through 1700, solve the given inequalities:

1698. $\dfrac{1+\sin x}{1-\cos x} + \dfrac{1-\sin x}{1+\cos x} \leqslant a.$ **1699.** $\dfrac{1+\sin x}{1+\cos x} + \dfrac{1-\sin x}{1-\cos x} \leqslant a.$

1700. $\cos x - \dfrac{1}{\cos x} \leqslant a.$

Answers

1. $(a-1)(a+1)(a^2+1)$. 2. $(a-1)(a+1)(a^2+a+1)(a^2-a+1)$. 3. $(a^2+1)(a^2+a\sqrt{3}+1)(a^2-a\sqrt{3}+1)$. 4. $(a-3)^2(a+3)^2$. 5. $(a-1)^2(a+1)^2(a^2+a+1)^2(a^2-a+1)^2$. 6. $(a-1)(a^2+1)(a^2+a+1)$. 7. $(a-1)(a+1)^3$. 8. $(a-b+c)(a+b-c)(-a+b+c)(a+b+c)$. 9. $(a^2+ab+b^2)(a^2-ab+b^2)$. 10. $(a+1)(a-1)(a^2+5)$. 11. $(a^2+1)(4a^2+1)$. 12. $(c-1)(c+1)(c^2-ab)$ 13. $(a^2+6a+18)(a^2-6a+18)$. 14. $(a^2+a+1)(a^2-a+1)$. 15. $(a^2+a+1)(a^2-a+1)(a^2+a\sqrt{3}+1)(a^2-a\sqrt{3}+1)$. 16. $(a^2+1)(2a^2+a+2)$. 17. $(a+\sqrt{2})\times(a-\sqrt{2})(a^2+3a+6)$. 18. $a(a+1)(a^2+a+7)$. 19. $(a-1)(a^2+1)(a^2+a+1)$. 20. $(a+2b)(2b-c)(a-c)$. 21. $(a+b)(b+c)(c+a)$. 22. $(a-2c)(b-2c)\times(a+b)$. 23. $a(a-1)(a+1)(a-2)(a+2)(a-3)(a+3)$. 24. $5ab(a+b)(a^2+ab+b^2)$. 25. $(a-b)(b-c)(c-a)(ab+bc+ca)$. 26. $(b+c)(2a-b)(2a+c)\times(2a+b-c)$. 27. $3(a+b)(b+c)(c+a)$. 28. $(a^2+a\sqrt{6}+3)\times(a^2-a\sqrt{6}+3)$. 29. $(a^2+ab\sqrt{2}+b^2)(a^2-ab\sqrt{2}+b^2)$. 30. $(a-1)(a+3)^2$. 31. $(a^2+3a+1)^2$. 32. $(a+2)(a+6)(a^2+8a+10)$. 33. $(3a^2+4a-1)(3a^2+2a+1)$. 34. $(a-b)(b-c)(a-c)(a+b+c)$. 35. $3(a-b)(b-c)(c-a)$. 36. $3(a+c)(a-c)(a^2+b^2)(b^2+c^2)$. 37. $(a^2-ab-b^2)(a^2+3ab+b^2)$. 38. $(a+b+c)(ab+bc+ac)$. 39. $(a+b+c)(a+b-c)(a-b+c)(a-b-c)$. 40. $(a+1)(a^2+a+1)(a^2-a+1)$. 41. $(a^2+a+1)^2$. 42. $(a+b)^2(a^2-4ab-b^2)$. 43. $(a^2+a\sqrt{2}+1)(a^2-a\sqrt{2}+2)$. 44. $(a^2+a+1)(a^8-a^7+a^5-a^4+a^3-a+1)$. 48. For $a=1$. 51. $\frac{5a+4}{a^2+a+1}$. 52. $\frac{a^4+1}{a+1}$. 53. $\frac{a^2-1}{a^4-2a^2+4}$. 54. $\frac{a^2-4}{a^2+5}$. 55. $\frac{2a^2+3}{3a^2-3}$. 56. $\frac{5a^2-3b}{a^2+2}$. 57. $\frac{1}{a^2-b^2}$. 58. $\frac{16a^{15}}{1-a^{16}}$. 59. $\frac{32}{1-a^{32}}$. 60. $\frac{5}{a(a+5)}$. 61. $\frac{a}{a^2-1}$. 62. $2a$. 63. $\frac{(a+b+c)^2}{2bc}$. 64. 0. 65. 0. 66. $\frac{1}{a+c}$. 67. $\frac{2a^4}{a^8-16b^8}$. 68. 0. 69. $a+b+c$. 70. $(a+b)\times(b+c)(c+a)$. 75. 9. 97. $S_n=\frac{n}{2n+1}$. 98. $S_n=\frac{n}{3n+1}$. 99. $S_n=\frac{n}{4n+1}$. 100. $S_n=\frac{n}{5n+1}$. 101. $S_n=(-1)^{n+1}\frac{n(n+1)}{2}$. 120. 39. 121. $\frac{56\sqrt{10}+12}{3}$. 122. 10. 123. $-\sqrt{xy}$. 124. $\frac{1}{b}$, if $-1\leqslant b\leqslant 1$ and b if $b<-1$; $b>1$. 125. $a+b$ if $a>0$ and $b>0$, and $\frac{a(a+b)}{b}$ if $a<0$ and

$b<0$. **126.** $2+\sqrt{3}$. **127.** $\sqrt{2}-1$. **128.** 3. **129.** $\sqrt{2}$. **130.** $(\sqrt{3}+1)\sqrt[4]{2}$. **131.** $1+\sqrt{2}$. **132.** $\sqrt{3}-1$. **133.** $\sqrt{5}-2$. **134.** $\frac{\sqrt{6}+\sqrt{2}}{2}$. **135.** $\frac{(\sqrt[4]{5}+\sqrt[4]{2})(\sqrt{5}+\sqrt{2})}{3}$.

136. $\frac{\sqrt[3]{225}+\sqrt[3]{105}+\sqrt[3]{49}}{8}$. **137.** $\frac{\sqrt{2}(\sqrt{3}+\sqrt{5})}{2}$.

138. $\frac{\sqrt{2}(1+\sqrt{2}-\sqrt{3})}{4}$. **139.** $\sqrt[3]{3}-\sqrt[3]{2}$. **140.** $\frac{\sqrt{21}-\sqrt{15}-\sqrt{14}+\sqrt{10}}{2}$.

141. $\frac{2-\sqrt[4]{8}}{2}$. **142.** $-\sqrt{\sqrt{2}+\sqrt[3]{2}}\,(\sqrt{2}-\sqrt[3]{3})\,(4+2\sqrt[3]{9}+3\sqrt[3]{3})$.

143. $1+\sqrt{2}$. **144.** True. **145.** True. **146.** True. **147.** True. **148.** True. **149.** True. **150.** True. **151.** True. **160.** $\frac{\sqrt{a}}{4}$. **161.** 2. **162.** $\sqrt{m}-\sqrt{n}$.

163. $\sqrt[3]{a}$. **164.** $\sqrt{a-1}$. **165.** -1, if $0<a\leqslant 1$; $-\frac{(\sqrt{1-a^2}+1)^2}{a^2}$ if $-1\leqslant a<0$. **166.** $\sqrt[3]{(m-n)^2}$. **167.** $a+b$. **168.** $4\sqrt{a}$. **169.** $\sqrt{\frac{a}{b}}$.

170. $\sqrt[3]{\frac{2a}{1+a}}$. **171.** $\frac{1}{1-a^2}$. **172.** $4a$. **173.** 1. **174.** $-\sqrt[6]{b}$.

175. $\frac{a^{\frac{2}{3}}+1}{4}$. **176.** $\frac{2\sqrt[4]{ab}}{\sqrt[4]{a}+\sqrt[4]{b}}$. **177.** $-\sqrt[6]{b}$. **178.** $\frac{a}{2}$. **179.** $a-1$ if $a>-1$, $a\neq 0$, $a\neq 1$, and $1-a$ if $a<-1$. **180.** $2a$. **181.** $\sqrt[6]{\frac{b}{a}}$. **182.** (a) 0; (b) 3; (c) -1. **183.** (a) $\frac{1}{\sqrt[3]{9}}$; (b) $\frac{1}{\sqrt[4]{125}}$. **184.** (a) 24; (b) 890. **185.** 0.

186. (a) $\log_3 12$; (b) $\frac{1}{3}$. **187.** (a) 0; (b) 0. **188.** 3.0970. **189.** $\frac{a+3}{2(a+1)}$.

190. $\frac{4(3-a)}{3+a}$. **191.** $\frac{b}{1-a}$. **192.** $\frac{2-a}{a+b}$. **193.** $\frac{1}{b}$. **194.** $\frac{a+2b-2}{1-a}$.

195. $\frac{3a-b+5}{a-b+1}$. **196.** $\frac{a+1}{2a+b}$. **197.** $\frac{r(p+q)}{pq}$. **198.** $\frac{5n-3}{6}$. **199.** 1.

207. $\log_a b$. **208.** $\log_a b$. **209.** $a+b$. **210.** $b^{\log_b a}$. **211.** $-a$ if $0<a<1$; $a-2$ if $a>1$. **212.** $\log_a b$ if $a>1$ and $b>1$ or if $0<a<1$ and $0<b<1$. **213.** 0 if $a>1$ and $b>1$ or if $0<a<1$ and $0<b<1$; $-2(\log_b a+\log_a b)$ if $a>1$ and $0<b<1$ or if $0<a<1$ and $b>1$. **214.** $\log_a b$. **215.** 2 if $1<a\leqslant b$; $2\log_a b$ if $1<b<a$. **269.** $a>b$. **270.** $a<b$. **271.** $a>b$. **272.** $a<b$. **273.** $a>b$. **274.** $a>b$. **275.** $a<b$. **276.** (a) $a=b$; (b) $a=b$. **277.** (a) $a>b$; (b) $a<b$. **278.** (a) $a>b$; (b) $a>b$. **279.** $a>b$. **280.** $a<b$. **281.** $a<b$. **282.** $a<b$. **283.** $a<b$. **284.** $d<b<a<c$. **295.** Yes. **296.** No. **297.** No. **298.** Yes. **299.** Yes. **300.** No. **301.** Yes. **302.** Yes. **303.** Yes. **304.** No. **305.** Yes. **306.** Yes. **307.** No. **308.** No.

309. Yes. **310.** No. **311.** No. **312.** No. **313.** No. **314.** 4. **315.** 0; -2. **316.** 13. **317.** 5; -2. **318.** 9. **319.** 8. **320.** 6. **321.** 6. **322.** 2; 34. **323.** 4. **324.** 3. **325.** 4. **326.** 8. **327.** 5. **328.** $5\frac{2}{3}$. **329.** 3; 4. **330.** 3. **331.** 1; -1; i; $-i$. **332.** 2; -2; $1+i\sqrt{3}$; $1-i\sqrt{3}$; $-1+i\sqrt{3}$; $-1-i\sqrt{3}$. **333.** $\sqrt{2}+i\sqrt{2}$; $\sqrt{2}-i\sqrt{2}$; $-\sqrt{2}+i\sqrt{2}$; $-\sqrt{2}-i\sqrt{2}$. **334.** i; $-i$; $\frac{\sqrt{3}+i}{2}$; $\frac{\sqrt{3}-i}{2}$; $\frac{-\sqrt{3}+i}{2}$; $\frac{-\sqrt{3}-i}{2}$. **335.** 1; $\frac{-1+i\sqrt{7}}{2}$; $\frac{-1-i\sqrt{7}}{2}$. **336.** -1; 2; 3. **337.** -1; -3; -5. **338.** $-\frac{2}{3}$; $-\frac{1}{2}$; 3. **339.** 1; 2; 5; $\frac{5}{2}$. **340.** -1; 2; $-3+i\sqrt{3}$; $-3-i\sqrt{3}$. **341.** 1; 2; $\frac{-1+i\sqrt{3}}{2}$; $\frac{-1-i\sqrt{3}}{2}$. **342.** 3; -3; -4; $i\sqrt{3}$; $-i\sqrt{3}$. **343.** -1; $\frac{-1+i\sqrt{15}}{4}$; $\frac{-1-i\sqrt{15}}{4}$. **344.** -1; 3; $\frac{1}{3}$. **345.** 4; $\frac{20+i\sqrt{59}}{17}$; $\frac{20-i\sqrt{59}}{17}$. **346.** 2; -2; $\frac{3\sqrt{21}}{7}$; $-\frac{3\sqrt{21}}{7}$. **347.** i; $-i$; $\frac{1+i\sqrt{23}}{4}$; $\frac{1-i\sqrt{23}}{4}$. **348.** $\frac{i\sqrt{6}}{2}$; $-\frac{i\sqrt{6}}{2}$; $1+2i$; $1-2i$. **349.** 2; -2; $2i$; $-2i$; $\frac{\sqrt{2}+i\sqrt{2}}{2}$; $\frac{\sqrt{2}-i\sqrt{2}}{2}$; $\frac{-\sqrt{2}+i\sqrt{2}}{2}$; $\frac{-\sqrt{2}-i\sqrt{2}}{2}$. **350.** 2; 3; $\frac{5+i\sqrt{3}}{2}$; $\frac{5-i\sqrt{3}}{2}$. **351.** 3; -1; $1+\sqrt{10}$; $1-\sqrt{10}$. **352.** 2; $\frac{1}{2}$; $\frac{1+2i\sqrt{6}}{5}$; $\frac{1-2i\sqrt{6}}{5}$. **353.** 0; 1; -1; -2. **354.** 0; 1; $\frac{1+i\sqrt{15}}{2}$; $\frac{1-i\sqrt{15}}{2}$. **355.** 1; 2; $\frac{9+i\sqrt{51}}{6}$; $\frac{9-i\sqrt{51}}{6}$. **356.** -1; 2; $1+i$; $1-i$; $\frac{-1+i\sqrt{7}}{2}$; $\frac{-1-i\sqrt{7}}{2}$. **357.** $\frac{3+\sqrt{21}}{2}$; $\frac{3-\sqrt{21}}{2}$; $\frac{3+i\sqrt{11}}{2}$; $\frac{3-i\sqrt{11}}{2}$. **358.** -3; 2; $\frac{-1+i\sqrt{15}}{2}$; $\frac{-1-i\sqrt{15}}{2}$. **359.** $\frac{-5+\sqrt{13}}{2}$; $\frac{-5-\sqrt{13}}{2}$; $\frac{-5+i\sqrt{3}}{2}$; $\frac{-5-i\sqrt{3}}{2}$. **360.** $-\frac{1}{2}$; $-\frac{5}{4}$; $\frac{-7+2i\sqrt{2}}{8}$; $\frac{-7-2i\sqrt{2}}{8}$. **361.** $\frac{11}{2}$; $\frac{9}{2}$; $\frac{10+i\sqrt{7}}{2}$; $\frac{10-i\sqrt{7}}{2}$. **362.** -3; -5; $-4+i\sqrt{7}$; $-4-i\sqrt{7}$. **363.** $-\frac{1}{2}$; $\frac{2+i}{5}$; $\frac{2-i}{5}$. **364.** 1; $-\frac{1}{2}$. **365.** $-\frac{1}{2}$; $\frac{3+2\sqrt{7}}{19}$; $\frac{3-2\sqrt{7}}{19}$. **366.** $-\frac{1}{2}$; $\frac{-1+i}{2}$; $\frac{-1-i}{2}$. **367.** $\frac{1}{2}$; $\frac{3}{2}$; $-\frac{1}{4}$. **368.** 0.3; 0.4; 0.5.

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369. $1; \frac{1}{2}; \frac{2}{3}$. 370. $-3; \frac{3+i\sqrt{83}}{4}; \frac{3-i\sqrt{83}}{4}$. 371. $\frac{5}{3}; \frac{-1+i\sqrt{7}}{2}; \frac{-1-i\sqrt{7}}{2}$. 372. $\frac{7}{4}; \frac{-2+3i\sqrt{2}}{4}; \frac{-2-3i\sqrt{2}}{4}$.

373. $\frac{1}{2}; \frac{-1+\sqrt{7}}{2}; \frac{-1-\sqrt{7}}{2}$. 374. $2; \frac{1}{2}; \frac{1+i\sqrt{3}}{2}; \frac{1-i\sqrt{3}}{2}$.

375. $2; \frac{1}{2}; \frac{-11+\sqrt{105}}{4}; \frac{-11-\sqrt{105}}{4}$. 376. $1; \frac{-3+\sqrt{5}}{2}; \frac{-3-\sqrt{5}}{2}$. 377. $2; 6; \frac{-3+i\sqrt{39}}{2}; \frac{-3-i\sqrt{39}}{2}$. 378. $\frac{3+\sqrt{5}}{2}; \frac{3-\sqrt{5}}{2}; \frac{-1+i\sqrt{3}}{2}; \frac{-1-i\sqrt{3}}{2}$. 379. $1+2i; 1-2i; \frac{-3+i\sqrt{11}}{2}; \frac{-3-i\sqrt{11}}{2}$. 380. $2; -1; \frac{-3+\sqrt{17}}{2}; \frac{-3-\sqrt{17}}{2}$.

381. $-1; -\frac{1}{4}; \frac{3+i\sqrt{17}}{8}; \frac{3-i\sqrt{17}}{8}$. 382. $2+i\sqrt{3}; 2-i\sqrt{3}; -2+i\sqrt{5}; -2-i\sqrt{5}$. 383. $-1; 9; \frac{5+i\sqrt{61}}{2}; \frac{5-i\sqrt{61}}{2}$.

384. $0; -1$. 385. $\frac{\sqrt{2}}{2}; -1+\frac{\sqrt{2}}{2}; -1-\frac{\sqrt{2}}{2}$. 386. $4; -4$. 387. $3; \frac{17}{19}$. 388. $\left(-\infty; \frac{7}{4}\right]$. 389. $\left(-\infty; \frac{5}{3}\right]$. 390. $\frac{3+\sqrt{5}}{2}; \frac{3-\sqrt{5}}{2}; \frac{3+i\sqrt{11}}{2}; \frac{3-i\sqrt{11}}{2}$. 391. $1+\sqrt{2}; 1-\sqrt{2}; 1+\sqrt{6}; 1-\sqrt{6}$; 392. $1; \frac{-3+\sqrt{17}}{2}$. 393. $-\sqrt{2}; 1-\sqrt{5}$. 394. $\frac{-5+\sqrt{113}}{4}$. 395. $\varnothing$.

396. $-2; 0$. 397. $-3; -2; 0; 1$. 398. $[-1; 0]$. 399. $-\frac{1}{2}; -\frac{5}{2}$.

400. $[2; +\infty)$. 401. $\frac{3}{2}; \frac{9}{2}$. 402. $[1, 2]$. 403. $1; \frac{11}{2}$. 404. $\frac{3}{2}$.

405. $2; \frac{2}{5}$. 406. -2. 407. $(-\infty; -2] \cup [2; \infty)$. 408. -2.

409. $\frac{7}{6}$. 410. $-3; 2; \frac{-1+\sqrt{65}}{2}$. 411. -1. 412. $(-\infty; -3] \cup [3; \infty)$.

413. $[-3; -2] \cup [2; 3]$. 414. 2. 415. $\frac{-1+\sqrt{5}}{2}$. 416. $\frac{1}{2}$. 417. $-\frac{2}{3}; \frac{1}{2}; 2$. 418. $(-4; -4), (-6; -2)$. 419. $(-4; -5), (5; 4)$. 420. $(2; -5), (-4; 3), \left(1+2\sqrt{3}; \frac{3+8\sqrt{3}}{3}\right), \left(1-2\sqrt{3}; \frac{3-8\sqrt{3}}{3}\right)$.

421. $(1; 1), \left(\frac{-5+\sqrt{140}}{5}; \frac{-7+\sqrt{140}}{7}\right), \left(\frac{-5-\sqrt{140}}{5}; \frac{-7-\sqrt{140}}{7}\right)$.

422. $(4;\ 1)$, $\left(-2+2i\sqrt{3};\ -\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$, $\left(-2-2i\sqrt{3};\ -\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$.
423. $(2;\ 0)$, $(0;\ -2)$. **424.** $(1;\ 1;\ 1)$, $(7;\ -3;\ -1)$. **425.** $(3;\ 1;\ 2)$, $\left(\frac{17}{9};\ \frac{53}{18};\ \frac{61}{18}\right)$. **426.** $\left(\frac{2}{3};\ 3\right)$, $\left(-\frac{2}{3};\ -3\right)$, $(1;\ 2)$, $(-1;\ -2)$.
427. $(1;\ 2)$, $\left(-\frac{239}{146};\ \frac{117}{146}\right)$. **428.** $(1;\ 1)$, $(3;\ 2)$, $(-1;\ -2)$, $(-11;\ -7)$.
429. $(1;\ 1;\ 1)$, $(-2;\ -2;\ -2)$. **430.** $(0;\ 0)$, $(4;\ 2)$, $(-2;\ -4)$. **431.** $(1;\ 0)$.
432. $(3;\ 1)$, $(3;\ -1)$, $\left(-\frac{5}{3};\ \frac{\sqrt{65}}{3}\right)$, $\left(-\frac{5}{3};\ -\frac{\sqrt{65}}{3}\right)$. **433.** $(3\sqrt{2};\ \sqrt{2})$, $(-3\sqrt{2};\ -\sqrt{2})$, $(3\sqrt{2};\ -\sqrt{2})$, $(-3\sqrt{2};\ \sqrt{2})$. **434.** $(3;\ 2)$, $(1;\ 4)$, $(-3;\ -4)$, $(-5;\ -2)$. **435.** $\left(\frac{1}{4};\ \frac{1}{6}\right)$, $\left(\frac{1}{12};\ \frac{1}{3}\right)$, $\left(-\frac{5}{24};\ -\frac{7}{24}\right)$, $\left(-\frac{3}{8};\ -\frac{1}{8}\right)$. **436.** $\left(0;\ \frac{\sqrt{3}}{3}\right)$, $\left(0;\ -\frac{\sqrt{3}}{3}\right)$, $(1;\ 1)$, $(-1;\ -1)$.
437. $(3;\ 1)$, $(-3;\ -1)$, $\left(\frac{14\sqrt{106}}{53};\ \frac{4\sqrt{106}}{53}\right)$, $\left(\frac{-14\sqrt{106}}{53};\ \frac{-4\sqrt{106}}{53}\right)$.
438. $\left(t;\ -\frac{3}{2}t\right)$, $t\in \boldsymbol{R}$. **439.** $(t;\ 7t)$, $t\in \boldsymbol{R}$. **440.** $\left(\frac{\sqrt{30}}{40};\ -\frac{\sqrt{30}}{10}\right)$, $\left(-\frac{\sqrt{30}}{40};\ \frac{\sqrt{30}}{10}\right)$, $\left(\frac{\sqrt{30}}{5};\ \frac{\sqrt{30}}{5}\right)$, $\left(-\frac{\sqrt{30}}{5};\ -\frac{\sqrt{30}}{5}\right)$. **441.** $(1;\ 3)$, $(-1;\ -3)$, $\left(\frac{8\sqrt{2}}{\sqrt{7}};\ -\frac{20\sqrt{2}}{\sqrt{7}}\right)$, $\left(-\frac{8\sqrt{2}}{\sqrt{7}};\ \frac{20\sqrt{2}}{\sqrt{7}}\right)$. **442.** $\left(2;\ \frac{1}{2}\right)$, $\left(-2;\ -\frac{1}{2}\right)$, $\left(\frac{\sqrt{10}}{5};\ -\frac{2\sqrt{10}}{5}\right)$, $\left(-\frac{\sqrt{10}}{5};\ \frac{2\sqrt{10}}{5}\right)$. **443.** $(1;\ 2)$, $(-1;\ -2)$, $(2;\ 1)$, $(-2;\ -1)$. **444.** $(2;\ 3)$, $(-2;\ -3)$, $(3;\ 2)$, $(-3;\ -2)$.
445. $(2;\ 3)$, $(3;\ 2)$, $\left(-1+i\sqrt{3};\ -\frac{3}{2}+\frac{3i\sqrt{3}}{2}\right)$, $\left(-1-i\sqrt{3};\ -\frac{3}{2}-\frac{3i\sqrt{3}}{2}\right)$, $\left(-\frac{3}{2}+\frac{3i\sqrt{3}}{2};\ -1+i\sqrt{3}\right)$, $\left(-\frac{3}{2}-\frac{3i\sqrt{3}}{2};\ -1,\ -i\sqrt{3}\right)$.
446. $(0;0)$, $(\sqrt{7};\ \sqrt{7})$, $(-\sqrt{7};\ -\sqrt{7})$, $(\sqrt{19};\ -\sqrt{19})$, $(-\sqrt{19};\ \sqrt{19})$, $(2;\ 3)$, $(-2;\ -3)$, $(3;\ 2)$, $(-3;\ -2)$.
447. $\left(\frac{\sqrt{2}+i\sqrt{2}}{2};\ \sqrt{2}+i\sqrt{2}\right)$, $\left(\frac{\sqrt{2}-i\sqrt{2}}{2};\ 2-i\sqrt{2}\right)$, $\left(\frac{\sqrt{-2}+i\sqrt{2}}{2};\ -2+i\sqrt{2}\right)$, $\left(\frac{-\sqrt{2}-i\sqrt{2}}{2};\ -2-i\sqrt{2}\right)$, $(2;\ 1)$, $(-2;\ -1)$, $(2i;\ i)$, $(-2i;\ -i)$. **448.** $(3;\ 1)$, $(1;\ 3)$, $(-1;\ -3)$, $(-3;-1)$.
449. $(3;\ 2)$, $(-2;\ -3)$, $(0;\ 0)$. **450.** $(0;\ 0)$, $(3;\ 2)$, $(-4;\ -12)$. **451.** $(3;\ 5)$, $(5;\ 3)$, $(-5+2i\sqrt{2};\ -5-2i\sqrt{2})$, $(-5\ \ -2i\sqrt{2};\ -5+2i\sqrt{2})$.
452. $(2;\ 3)$, $(3;\ 2)$, $(-2+\sqrt{7};\ -2-\sqrt{7})$, $(-2-\sqrt{7};\ -2+\sqrt{7})$.
453. $(-2;\ 3)$, $(3;\ -2)$. **454.** $(6;\ 6)$, $\left(\frac{-3+3\sqrt{5}}{2};\ \frac{-3-3\sqrt{5}}{2}\right)$,

$\left(\frac{-3-3\sqrt{5}}{2};\ \frac{-3+3\sqrt{5}}{2}\right)$. **455.** (1; 4), (4; 1), $\left(\frac{-5+\sqrt{41}}{2};\ \frac{-5-\sqrt{41}}{2}\right)$, $\left(\frac{-5-\sqrt{41}}{2};\ \frac{-5+\sqrt{41}}{2}\right)$. **456.** (2; 3), (3; 2), $\left(-\frac{3}{4}+\sqrt{\frac{103}{48}};\ -\frac{3}{4}-\sqrt{\frac{103}{48}}\right)$, $\left(-\frac{3}{4}-\sqrt{\frac{103}{48}};\ -\frac{3}{4}+\sqrt{\frac{103}{48}}\right)$. **457.** (2; 3), (3; 2). **458.** (1; 5), (5; 1), (−1; −5), (−5; −1). **459.** (2; 1), (1; 2). **460.** (2; 1), (1; 2), (−2; −1), (−1; −2); (0; 0). **461.** (2; −1), (−2; 1), (1; −2), (−1; 2). **462.** (3; 1), (−1; −3), $(\sqrt[3]{13};\ -\sqrt[3]{13})$. **463.** (1; 2; −1), (−1; −2; 1). **464.** (1; 2; −1), (−1; −2; 1), $\left(\frac{3\sqrt{7}}{7};\ \frac{5\sqrt{7}}{7};\ -\frac{\sqrt{7}}{7}\right)$, $\left(-\frac{3\sqrt{7}}{7};\ -\frac{5\sqrt{7}}{7};\ \frac{\sqrt{7}}{7}\right)$. **465.** (2; 1; 0), (−2; −1; 0). **466.** (0; 0; 0), (1; 1; 1), $(0;\ \sqrt{2};\ \sqrt{2})$, $(0;\ -\sqrt{2};\ -\sqrt{2})$, $(\sqrt{2};\ 0;\ \sqrt{2})$, $(-\sqrt{2};\ 0;\ -\sqrt{2})$, $(\sqrt{2};\ \sqrt{2};\ 0)$, $(-\sqrt{2};\ -\sqrt{2};\ 0)$. **467.** (1; 2; 3), (1; 4; 1), (5; 2; −1), (5; 4; −3). **468.** (1; −2; 3), (3; −2; 1), (1; −3; 2), (2; −3; 1), (5; −1; 1), (1; −1; 5). **469.** (0; 0; 0), $(\sqrt{2};\ \sqrt{2};\ \sqrt{2})$, $(-\sqrt{2};\ -\sqrt{2};\ -\sqrt{2})$. **470.** (3; 2; 5), (3; −2; −5), (−3; −2; 5), (−3; 2; −5). **471.** (9; 3; 1), (1; 3; 9). **472.** (3; −2; 2), $\left(\frac{9+3\sqrt{5}}{2};\ \frac{-7-3\sqrt{5}}{2};\ \frac{1-3\sqrt{5}}{2}\right)$, $\left(\frac{9-3\sqrt{5}}{2};\ \frac{-7+3\sqrt{5}}{2};\ \frac{1+3\sqrt{5}}{2}\right)$. **473.** (1; 2; 3). **474.** (1; 1; 1). **475.** (1; 1; 1). **476.** $\left(1;\ 2;\ \frac{1}{2}\right)$, $\left(2;\ \frac{1}{2};\ 1\right)$, $\left(\frac{1}{2};\ 1;\ 2\right)$, $\left(1;\ \frac{1}{2};\ 2\right)$, $\left(2;\ 1;\ \frac{1}{2}\right)$, $\left(\frac{1}{2};\ 2;\ 1\right)$. **477.** $\left(\frac{7-\sqrt{113}}{2};\ \frac{7+\sqrt{113}}{2};\ 9\right)$, $\left(\frac{7+\sqrt{113}}{2};\ \frac{7-\sqrt{113}}{2};\ 9\right)$, $\left(\frac{-7+\sqrt{113}}{2};\ \frac{-7-\sqrt{113}}{2};\ -9\right)$, $\left(\frac{-7-\sqrt{113}}{2};\ \frac{-7+\sqrt{113}}{2};\ -9\right)$, (3; 4; 5), (4; 3; 5), (−3; −4; −5), (−4; −3; −5). **478.** (0; 0; 0), (1; 2; 1), (2; 1; 1), $\left(\frac{3+\sqrt{6}}{3};\ \frac{3-\sqrt{6}}{3};\ \frac{2}{3}\right)$, $\left(\frac{3-\sqrt{6}}{3};\ \frac{3+\sqrt{6}}{3};\ \frac{2}{3}\right)$.
479. (13; 0; 13), (8; 2; 4). **480.** 31. **481.** 24. **482.** 12 and 1232. **483.** 103. **484.** 285 714. **485.** 54. **486.** 83. **487.** 428 and 824. **488.** 8 hours. **489.** 820. **490.** 6; $-\frac{1}{2}$. **491.** 12; 24; 36; 54 or 52.5; 37.5; 22.5; 13.5. **492.** 5103 or $\frac{7}{81}$. **493.** 931. **494.** 1350. **495.** 12, 18, 27. **496.** 20. **497.** 5. **498.** 0.25 kg. **499.** Either 12 or 9.5 roubles. **500.** 2. **501.** 24 and 16. **502.** 35 kg of wheat-flour and 45 kg of rye-flour. **503.** By 20%. **504.** 3%. **505.** 200 roubles. **503.** By 33.8%. **507.** By 10%. **508.** 1. **509.** 44 workers. **510.** 32 students. **511.** 20 km. **512.** 50 km/h. **513.** 10 km/h. **514.** Either 360 cm and 18 cm/s or 60 cm and 6 cm/s. **515.** 1375 km. **516.** 840 km, 80 km/h, 70 km/h. **517.** 40 m/min. **518.** 6 km/h and 3 km/h. **519.** 6 m/s and 8 m/s. **520.** 20 km/h. **521.** 20 km/h. **522.** 3 km/h and 1 km/h. **523.** 8 km. **524.** 10 hours and 9 hours. **525.** 60 km/h and 40 km/h. **526.** 15 hours and 10 hours. **527.** $a(1+\sqrt{2})$ hours. **528.** 50 km/h and 100 km/h. **529.** 60 km/h and 100 km/h. **530.** 40 m/s and 36 m/s. **531.** 15 m/s, 10 m/s. **532.** 20 m/min, 15 m/min, 280 m. **533.** $\frac{1}{80}$, $\frac{1}{90}$. **534.** 25 hours.

535. 16 hours. **536.** 2 hours. **537.** 25 km/h. **538.** The speeds of the steam-launches are equal to 15 km/h, the rate of flow of the river is equal to 3 km/h. **539.** 14 km/h. **540.** 1 s. **541.** 10 km/h. **542.** 20 km/h. **543.** The speed of the first pedestrian is twice the speed of the second. **544.** At 8.30 p.m. **545.** Ten-fold. **546.** 3 hours. **547.** 1:2. **548.** $30 < v < 40$. **549.** 8 km/h and 7 km/h. **550.** With the cyclist. **551.** 2.75. **552.** 48 km/h. **553.** 6 hours and 4 hours. **554.** 4 hours. **555.** 3 hours. **556.** 24 hours. **557.** 90 s. **558.** 20/33 hours and $\frac{10}{3}$ hours. **559.** 4 hours and $\frac{4}{3}$ hours. **560.** 80 km/h. **561.** 60%. **562.** The first team manufactured 13 workpieces, the second team 11. **563.** 60 m^3/h and 24 m^3/h. **564.** 16 hours. **565.** Twice as much delivers the second pipe. **566.** The oil-level rose. **567.** 20 hours and 30 hours. **568.** 3 hours and 4 hours. **569.** 12 hours and 8 hours. **570.** $\frac{5}{6}$ hour and $\frac{5}{18}$ hour. **571.** 16 days. **572.** $\frac{20}{3}$ hours and $\frac{16}{3}$ hours. **573.** 7.5 hours and 10.5 hours. **574.** 14.4 hours. **575.** 3 hours. **576.** The productivity of the second factory is twice the productivity of the first. **577.** 6 days. **578.** $\frac{60}{7}$ minutes. **579.** 8 hours. **580.** 50 hours. **581.** Three-fold. **582.** $\frac{6}{5}$ times. **583.** 10 days. **584.** 28 roubles and A is more expensive. **585.** 300 g and 500 g. **586.** 441 g. **587.** 40 tonnes and 60 tonnes. **588.** 187.5 kg. **589.** 15 tonnes. **590.** 53%. **591.** 5%. **592.** 10 kg. **593.** $\simeq 2.77$ kg. **594.** 1:3. **595.** 1.64 litres and 1.86 litres. **596.** 15 kg. **597.** 10 kg, 69%. **598.** Two times. **599.** 18 kg. **600.** $a+b-c$. **601.** 6 litres. **602.** 18 litres. **603.** 2.4 kg and 4.8 kg. **604.** 3.5 litres of glycerin and 0.5 litre of water. **605.** 10 litres. **606.** 5% and 10%. **607.** 15% and 40%. **608.** 62.5% and 55%. **609.** $\varnothing$. **610.** 7; 8. **611.** 2. **612.** 0. **613.** 0; 2. **614.** $\varnothing$. **615.** $2\sqrt{2}$; $-2\sqrt{2}$. **616.** 0; 0.5. **617.** 1.25. **618.** 1; -1. **619.** 64. **620.** 1; $-\frac{1}{3}$. **621.** 1; $-\frac{8}{3}$. **622.** 1; 2. **623.** 4; -4. **624.** 2. **625.** 1024. **626.** 1. **627.** -0.5. **628.** 1. **629.** 6; -2. **630.** 2; -7. **631.** 4; -1. **632.** $\frac{-1+\sqrt{74602}}{18}$; $\frac{-1-\sqrt{74602}}{18}$. **633.** 1. **634.** -1; 8; 27. **635.** 1; $-\frac{1}{8}$. **636.** 1. **637.** -1; 0. **638.** -2; 1. **639.** 5. **640.** -37; 6. **641.** 2. **642.** 15. **643.** -2; 5. **644.** 1. **645.** 2. **646.** 1. **647.** -88; -24; 3. **648.** -1; $-\frac{1}{2}$; 1; 2. **649.** 2. **650.** 1. **651.** 1; 2; 10. **652.** 1; 20. **653.** -3; 3. **654.** -2. **655.** 8; $8+\frac{12\sqrt{21}}{7}$; $8-\frac{12\sqrt{21}}{7}$. **656.** 0. **657.** 1416. **658.** 9. **659.** 12. **660.** 1. **661.** 2; 3. **662.** 1; 4. **663.** 2; 6. **664.** -61; 4. **665.** 16; 81. **666.** 2; 6. **667.** 1; 32. **668.** $17+\sqrt{257}$; $17-\sqrt{257}$. **669.** $\frac{6\sqrt{119}}{119}$. **670.** 0. **671.** 0.25. **672.** 3; $\frac{5+\sqrt{297}}{8}$. **673.** 2; 3. **674.** 1; -6. **675.** 1. **676.** 3. **677.** 5. **678.** $\frac{841}{144}$. **679.** (1; 4). **680.** (9; 4). **681.** (3; 2), $\left(\frac{17}{27}; -\frac{14}{9}\right)$. **682.** $\left(\frac{10}{3}; -\frac{7}{3}\right)$. **683.** (7; 13), $(-7; -13)$,

$\left(\frac{13}{2};\ 14\right)$, $\left(-\frac{13}{2};\ -14\right)$. **684.** (3; 1). **685.** (12; 4), (34; −30), $(103-19\sqrt{17};\ -77+25\sqrt{17})$. **686.** (2; 8), (8; 2). **687.** (1; 9),(9; 1). **688.** (4; 1), (1; 4). **689.** (8; 1), (1; 8). **690.** (8; 1), (1; 8). **691.** (8; 1), (−8; 1), (−8; −1), (8; −1). **692.** (1; 7), (7; −8), $\left(\frac{49}{64};\ \frac{41}{8}\right)$. **693.** (0; 0). **694.** (5; 4). **695.** (2; 3), $\left(\frac{13}{3};\ -\frac{5}{3}\right)$. **696.** $\left(\frac{17}{12};\ \frac{5}{3}\right)$ **697.** (4; 9; 1), (−4; −9; −1). **698.** (3; −2; 6). **699.** (5; 4; 5). **700.** $\left(\frac{9}{58};\ -\frac{6}{29};\ \frac{33}{29}\right)$. **701.** 3. **702.** −3; 1. **703.** $\frac{3+\sqrt{13}}{2}$; $\frac{3-\sqrt{13}}{2}$. **704.** 3; $-\frac{5}{2}$. **705.** 3; $-\frac{1}{5}$. **706.** 3. **707.** 2. **708.** 0. **709.** 0. **710.** −1; 1; $\sqrt{2}$; $-\sqrt{2}$. **711.** 3. **712.** 1. **713.** 3. **714.** 2.5. **715.** 0. **716.** 1.5. **717.** −1; 1. **718.** 0. **719.** $\log_{0.4}5$; −1. **720.** 0. **721.** −1; 1. **722.** 1. **723.** 0. **724.** 0. **725.** 1; $1+\sqrt{2}$; $1-\sqrt{2}$. **726.** 1; $-2(1+\log_3 2)$. **727.** 2; $-1-\log_5 2$. **728.** −1; 1; 2. **729.** −1; 1; 2. **730.** −3; 1; 2; 3; 4. **731.** $\frac{4}{3}$; $\frac{5}{3}$; 2. **732.** 2. **733.** 2. **734.** $\sqrt{10}$; 9. **735.** −1; 1; 4. **736.** 2; 3. **737.** 0; 1.5. **738.** $2-\sqrt{3}$; $2+\sqrt{3}$. **739.** 0.5. **740.** 1.5; 3. **741.** 8. **742.** 6. **743.** 1; 2. **744.** 1.5; 10. **745.** 37. **746.** $\frac{5}{28}$. **747.** 3; −5. **748.** 2; 3. **749.** $\varnothing$. **750.** 5. **751.** 4; 6. **752.** 41. **753.** 0.75. **754.** 3. **755.** 2. **756.** $\varnothing$. **757.** 1. **758.** 4. **759.** 2. **760.** 2. **761.** 2; 8. **762.** $\frac{3-2\sqrt{2}}{8}$; $\frac{3+2\sqrt{2}}{8}$. **763.** −4. **764.** 10^{-3}; 10^{-1}; 10, 10^3. **765.** $-1+\sqrt{10}$; 9. **766.** 10. **767.** 10^{-1}; 10^5. **768.** 10^{-1}; $10^{\frac{1}{4}}$. **769.** $\sqrt[5]{5}$; 5. **770.** 10. **771.** $\sqrt{2}$; 4. **772.** 2^{-7}; 2. **773.** −8; 1.9. **774.** −5; 5. **775.** $\frac{5}{12}$. **776.** $\frac{1}{3\sqrt[3]{3}}$; $\frac{1}{\sqrt{3}}$. **777.** 1. **778.** 2. **779.** 10; 10^5. **780.** 0.5; 32. **781.** 10^{-1}; 10^3. **782.** 10^{-2}; 10^2. **783.** $\varnothing$. **784.** 10^{-4}; 10. **785.** 5^{-1}; 5^2. **786.** 3^{-1}; 3^2. **787.** $\sqrt{626}$. **788.** 2. **789.** $\frac{1}{3}$; $\frac{1}{15}$. **790.** 2^{-4}; 2. **791.** 2^{-1}; 1; 16. **792.** 100. **793.** 1; $10^{-\sqrt{\log 1.5}}$; $10^{+\sqrt{\log 1.5}}$. **794.** 9^{-1}; 9. **795.** $\frac{1}{\sqrt[4]{8}}$; 1; 2. **796.** $2^{-\frac{1}{9}}$. **797.** 3. **798.** 0. **799.** 17. **800.** 10^{-1}; 2; 10^3. **801.** −2. **802.** 7; 14. **803.** 1. **804.** −1; $\frac{1-\sqrt{46}}{5}$;

$\frac{1+\sqrt{46}}{5}$. **805.** 1; 2. **806.** (—10; —12), (12; 10). **807.** (2; 3), (3; 2). **808.** $\left(\frac{1}{6}; \frac{1}{4}\right)$, $\left(\frac{1}{4}; \frac{1}{6}\right)$. **809.** $\left(\frac{1}{2}; \frac{\sqrt{2}}{5}\right)$ **810.** (3; 2). **811.** (4; 1). **812.** (3; 2). **813.** (1; 1), (4; 2). **814.** (1; 1), (2; 4), (—2; 4). **815.** (1; 2), (2; 1). **816.** $\left(\frac{1}{4}; 64\right)$, (8; 2). **817.** (9; 7). **818.** (2; 32), (32; 2). **819.** $\left(\frac{1}{3}; \frac{7}{3}\right)$, (3; 1). **820.** (7; 3). **821.** (17; 9). **822.** (2; 6). **823.** (125; 4), (625; 3). **824.** (3; 27), (27; 3). **825.** $\left(-\frac{7\sqrt{2}}{2}; \frac{\sqrt{2}}{2}\right)$. **826.** (4; 1). **827.** (1; 1), $(3\sqrt{3}; \sqrt{3})$. **828.** $(9; \sqrt[3]{9})$, $(\sqrt[3]{9}; 9)$. **829.** $\left(\frac{1}{8}; 64\right)$, $\left(\frac{1}{2}; \frac{1}{4}\right)$. **830.** (5.5; 2.5). **831.** (2; 3), (t; 1), where $1 < t < 3$. **832.** $\left(\sqrt{\frac{\sqrt{5}+1}{2}}; \sqrt{\frac{\sqrt{5}-1}{2}}\right)$. **833.** $(\log_4 12; \log_4 3)$. **834.** (1; —1). **835.** $(0; 1) \cup (1; \infty)$. **836.** $\left(-\infty; -\frac{1}{3}\right) \cup \left(\frac{3}{2}; 2\right)$. **837.** $(-\infty; -2) \cup (-2; -1) \cup \left(\frac{2}{3}; 3\right)$. **838.** (—5; 5). **839.** $(-\infty; -8) \cup (0; 8)$. **840.** [2; 5]. **841.** $\left(\frac{7-\sqrt{61}}{2}; \frac{7+\sqrt{61}}{2}\right)$. **842.** 4. **843.** $x \in \boldsymbol{R}$. **844.** $(-\infty; -6) \cup [-2; \infty)$. **845.** $(-\infty; 1)$. **846.** —1; 1. **847.** $\left(-\infty; \frac{2}{3}\right) \cup \left(1; \frac{5}{2}\right)$. **848.** $(-4; -3) \cup (-2; -1) \cup \left(\frac{1}{2}; 3\right)$. **849.** $(-\infty; 4] \cup \left[\frac{1-\sqrt{13}}{2}; \frac{1+\sqrt{13}}{2}\right] \cup [4; \infty)$. **850.** $(-2; 2) \cup (2; 4)$. **851.** $\left[-3; \frac{1-\sqrt{41}}{4}\right] \cup \left[0; \frac{1+\sqrt{41}}{4}\right] \cup \{3\}$. **852.** $(-\infty; 2) \cup (3; 5) \cup$ 7; ∞). **853.** $(-\infty; -3) \cup (2-\sqrt{6}; 3) \cup (2+\sqrt{6}; \infty)$. **854.** $\left(-\frac{5}{3}; 0\right] \cup \left(\frac{5}{3}; \infty\right)$. **855.** (—1; 5). **856.** (—8, 1]. **857.** $\left(-2; \frac{-1-\sqrt{5}}{2}\right) \cup \left(\frac{-1+\sqrt{5}}{2}; 2\right)$. **858.** $\left(-\infty; \frac{3}{2}\right) \cup \left(\frac{7}{3}; \infty\right)$. **859.** $(-\infty; -2) \cup [1; \infty)$. **860.** $(-\infty; 0) \cup (3; \infty)$. **861.** $(-\infty; -1)$. **862.** $(-3; -2) \cup (-1; 1)$. **863.** $(-\infty; 2) \cup (2; \infty)$. **864.** $\left(\frac{1-\sqrt{73}}{6}; -1\right) \cup \left(1; \frac{1+\sqrt{73}}{6}\right)$. **865.** $(-\infty; 1) \cup \left(\frac{4}{3}; 2\right)$. **866.** $\left(-\infty; -\frac{4}{3}\right) \cup \left(-\frac{79}{75}; \frac{3}{2}\right) \cup (2; \infty)$. **867.** $(-\infty; -7] \cup (-1; 0) \cup [0; 1] \cup (3; \infty)$. **868.** $(-\infty; -1) \cup (-1; 2]$. **869.** $(-\infty; 2) \cup [3.5; 4) \cup (7; \infty)$. **870.** $(-\infty; 5)$. **871.** (0. 9). **872.** (2.7; 6). **873.** (1; 2).

874. $[1;\ 2)$. **875.** $(-3;\ -\sqrt{7}) \cup (\sqrt{7};\ 3)$. **876.** $\left(-\infty;\ \frac{2}{3}\right) \cup \left(\frac{7}{4};\ 2\right)$. **877.** $(-1;\ 1) \cup (3;\ 5)$. **878.** $\left(-5;\ -\frac{3}{2}\right) \cup \left(\frac{1}{2};\ 1\right)$. **879.** $\left[-\frac{1}{2};\ \frac{1}{2}\right]$. **880.** $(0;\ 1)$. **881.** $(-4;\ -3) \cup [-2;\ -1] \cup [1;\ 2)$. **882.** $(-8;\ -6.5) \cup (0;\ 5)$. **883.** $(-4;\ -3) \cup (-2;\ -1) \cup \left(0;\ \frac{1}{3}\right) \cup (1;\ 2) \cup (2;\ 3) \cup (3;\ 4)$. **884.** $(-\infty;\ -7) \cup (-7;\ -2] \cup (1;\ 7) \cup (7;\ 8] \cup (11;\ \infty)$. **885.** $\left[-1;\ -\frac{1}{2}\right] \cup \left[\frac{1}{2};\ 1\right]$. **886.** $(-\infty;\ 1) \cup (1;\ 2) \cup (2;\ \infty)$. **887.** $\left[\frac{37}{7};\ 6\right) \cup (6;\ 7]$. **888.** $\left[\frac{5}{3};\ 2\right] \cup (3;\ \infty)$. **889.** $\varnothing$. **890.** $(-\infty;\ 1) \cup (2;\ 3)$. **891.** $\left(-\infty;\ \frac{2}{3}\right) \cup (3;\ \infty)$. **892.** $(-1;\ 1)$. **893.** $[0;\ 8]$. **894.** $(-\infty;\ 2) \cup (3;\ 7)$. **895.** (a) $(1;\ 6)$; (b) $\left(\frac{5}{9};\ 1\right] \cup [6;\ \infty)$; (c) $\varnothing$. **896.** $(-\infty;\ -1) \cup (2;\ 3)$. **897.** $(-6;\ 2)$. **898.** $(-2;\ 4)$. **899.** $(-\infty;\ -16) \cup (6;\ \infty)$. **900.** $(1;\ 4)$. **901.** $\left(-\infty;\ -\frac{4}{3}\right] \cup [2;\ \infty)$. **902.** $[1.5;\ 2.5]$. **903.** $(-\infty;\ -1) \cup (0;\ \infty)$. **904.** $(-\infty;\ -0.4] \cup [4;\ \infty)$. **905.** $(-\infty;\ -5) \cup (-1;\ 1) \cup (1;\ \infty)$. **906.** $(-\infty;\ -2) \cup \left(\frac{4}{3};\ \infty\right)$. **907.** $[4.5;\ \infty)$. **908.** $(-\infty;\ 1] \cup [1.5;\ \infty)$. **909.** $x \in \boldsymbol{R}$. **910.** $(0;\ 0.4)$. **911.** $\left[0;\ \frac{1}{3}\right]$. **912.** $\left(-\infty;\ \frac{3-\sqrt{65}}{4}\right) \cup \left(\frac{3-\sqrt{33}}{4};\ \frac{3+\sqrt{33}}{4}\right) \cup \left(\frac{3+\sqrt{65}}{4};\ \infty\right)$. **913.** $(-\infty;\ 1) \cup (2.2;\ \infty)$. **914.** $\left[\frac{-1-\sqrt{11}}{2};\ -1\right) \cup (-1;\ 1) \cup \left[1;\ \frac{-1+\sqrt{11}}{2}\right]$. **915.** $(-\infty;-2) \cup (-2;\ -1) \cup (-1;\ 0)$. **916.** $(-\infty;\ -2] \cup [-1;\ \infty)$. **917.** $(-\infty;\ -2) \cup (-2;\ 0] \cup [1.6;\ 2) \cup (2;\ 2.5]$. **918.** $[-1;\ 1]$. **919.** $(-\infty;\ -3) \cup (3;\ \infty)$. **920.** $\left(-\infty;\ -\frac{5}{3}\right) \cup (3;\ \infty)$. **921.** $(-\infty;\ -4] \cup [1;\ \infty)$. **922.** $(-\infty;\ -5) \cup (-1;\ \infty)$. **923.** $x \in \boldsymbol{R}$. **924.** $[1.5;\ 2)$. **925.** $(1;\ 3)$. **926.** $(-5;\ -2) \cup (2;\ 3) \cup (3;\ 5)$. **927.** $(-\infty;\ 3)$. **928.** $(-2;\ 3)$. **929.** $(-\infty;\ -2) \cup (3;\ \infty)$. **930.** $\left[\frac{2}{7};\ \frac{2}{3}\right]$. **931.** $\left(-\frac{1}{2};\ \frac{11}{4}\right)$. **932.** $(-\infty;\ 0) \cup (6;\ \infty)$. **933.** $(-\infty;\ -4) \cup (-2;\ 1) \cup (3;\ \infty)$. **934.** $(-\infty;\ 2] \cup [4;\ \infty)$. **935.** $(-\infty;\ 2)$ **936.** $(0;\ \infty)$. **937.** $\left[\frac{1}{3};\ 3\right]$. **938.** $\left(-\infty;\ \frac{7}{4}\right) \cup \left(\frac{5}{2};\ \infty\right)$.

939. There are 29 parts in the first box and 7 in the second. **940.** There are 11 workers in the first team and 17 workers in the second. **941.** 119. **942.** 25,300 m. **943.** 850 litres. **944.** 9 persons in each team. **945.** 8 books. **946.** 11 "twos", 7 "threes", 10 "fours", and 2 "fives". **947.** 180 roubles. **948.** 14 roubles. **949.** $[-0.5;\ 12)$. **950.** $(1;\ \infty)$.

951. $[2.6; 4)$. **952.** $(-\infty; 0.5] \cup [0.68; \infty)$. **953.** $(3; \infty)$. **954.** $(-\infty; -1)$.
955. $[0.5; \infty)$. **956.** $(-\infty; -2] \cup \left(5; 5\frac{9}{13}\right)$. **957.** $[4; \infty)$.
958. $(-3; 1)$. **959.** $\left[\frac{20}{9}; 4\right) \cup (5; \infty)$. **960.** $(-\infty; 0] \cup (4.5; \infty)$.
961. $(-\infty; 0]$. **962.** $\left[-\frac{10}{13}; 2\right] \cup [3; \infty)$. **963.** $(-\infty; \infty)$.
964. $[3; \infty)$. **965.** $\left[4; 4\frac{9}{16}\right]$. **966.** $\varnothing$. **967.** $[4; 5)$.
968. $\left[3; \frac{15+16\sqrt{15}}{15}\right]$. **969.** $\varnothing$. **970.** $\varnothing$. **971.** $\left[2.5; \frac{-5+\sqrt{149}}{2}\right)$.
972. $\varnothing$. **973.** $[\sqrt{21}; 2\sqrt{7}]$. **974.** $(-5; 5)$. **975.** $\left(2; \frac{4\sqrt{3}}{3}\right]$.
976. $(9; \infty)$. **977.** $(-\infty; -2) \cup (20.5; \infty)$. **978.** $(-\infty; -4) \cup (1; \infty)$.
979. $[-1; 4]$. **980.** $(-1; 3] \cup [3.5; 7.5)$. **981.** $(2; \infty)$. **982.** $(-\infty; \infty)$.
983. $(-\infty; \infty)$. **984.** $(0; \infty)$. **985.** $[2; 6]$. **986.** $(-\infty; \sqrt[3]{2}) \cup (\sqrt[3]{2}; \infty)$.
987. $(-\infty; -2) \cup (0; 1) \cup (1; \infty)$. **988.** $(-2; -1] \cup \left[-\frac{2}{3}; \frac{1}{3}\right)$.
989. $(-\infty; -4+2\sqrt{5})$. **990.** $\left(-\infty; -\frac{13}{6}\right] \cup [3; \infty)$. **991.** $(2; 8)$.
992. $[-2; 0) \cup (0; 2]$. **993.** $(5; \infty)$. **994.** $[-1; \infty)$. **995.** $(-\infty; -2] \cup$ $\left[-1; \frac{\sqrt{13}-1}{6}\right)$. **996.** $[0; 3]$. **997.** $[2; 5]$. **998.** $[-1; 0]$. **999.** $(0, \infty)$.
1000. $(-\infty; 0.4)$. **1001.** $(-\infty; 1.5)$. **1002.** $(-\infty; -1) \cup (7; \infty)$.
1003. $(-\infty; -6] \cup [2; \infty)$. **1004.** $(-\infty; 1-\log_2 3)$. **1005.** $(-2\sqrt{2}; 2\sqrt{2})$. **1006.** 1; 2; 3; 4; 5; 6; 7. **1007.** $[-3; -\sqrt{6}) \cup (-\sqrt{6}; -2] \cup [2; \sqrt{6}) \cup (\sqrt{6}; 3]$. **1008.** $(-1; 0) \cup (0; 1) \cup (1; 2)$. **1009.** $\left(-2; -\frac{5}{3}\right) \cup \left(0; \frac{1}{3}\right)$. **1010.** $(-\infty; 66]$. **1011.** $\left(\frac{2}{3}; \log_8 60\right]$. **1012.** $(2; \infty)$.
1013. $(3; \infty)$. **1014.** $(0; \infty)$. **1015.** $(-1; 1)$. **1016.** $(0; 1)$.
1017. $(2; \infty)$. **1018.** $\varnothing$. **1019.** $(0; \infty)$. **1020.** $(-\infty; \log_{1.5} 0.5)$.
1021. $(0; \infty)$. **1022.** $(-\infty; \log_2(1+\sqrt{3})]$. **1023.** $\left(-\frac{1}{3}; \infty\right)$.
1024. $(2; \infty)$. **1025.** $(0; 2)$. **1026.** $\left(\sqrt{\frac{7}{2}}; 2\right)$. **1027.** $\left(-\infty; -\frac{1}{2}\right) \cup (1; \infty)$. **1028.** $[\log_5 7; 2]$. **1029.** $[\log_{13} 5; 1]$. **1030.** $(0; 0.5)$.
1031. $(1; 1.5)$. **1032.** $(1; 2) \cup (4; 5)$. **1033.** $(-1, 0) \cup (1; 2)$.
1034. $(-1; 1) \cup (3; 5)$. **1035.** $(4; 5) \cup [95; \infty)$.
1036. $\left(\frac{-1+2\sqrt{91}}{5}; 4\right)$. **1037.** $(3; 4.5)$. **1038.** $\left(-1; \frac{91}{9}\right)$.
1039. $(3; 4) \cup (4; \infty)$. **1040.** $(0; \infty)$; **1041.** $(1; 1.04) \cup (26; \infty)$. **1042.** $(3; 7)$.
1043. $\left(-2; \frac{13}{6}\right)$. **1044.** $[1; 4]$. **1045.** $(-\infty; -2) \cup (6; \infty)$.

1046. $\left(0; \frac{3-\sqrt{5}}{2}\right) \cup \left(\frac{3+\sqrt{5}}{2}; 3\right)$. **1047.** $\varnothing$. **1048.** $(1; 3)$. **1049.** $(1; 3)$. **1050.** $(0; 1) \cup \left[\frac{\sqrt{113}-7}{2}; 2\right)$. **1051.** $(-2\sqrt{3}; -2) \cup (2; 2\sqrt{3})$. **1052.** $(1; \infty)$. **1053.** $(0; 0.75) \cup (1.25; 2)$. **1054.** $[2; 3) \cup (3; 4]$. **1055.** $(1; 4)$. **1056.** $(2; \infty)$. **1057.** $(0; 1) \cup \left(\frac{1+\sqrt{5}}{2}; 2\right)$. **1058.** $(-\infty; 0) \cup (5; \infty)$. **1059.** $(-\infty; -7) \cup (-5; -2] \cup [4; \infty)$. **1060.** $[0.5; 4]$. **1061.** $(0; 0.5) \cup [2\sqrt{3}; \infty)$. **1062.** $(-\infty; -5) \cup (3; \infty)$. **1063.** $\left[\frac{1}{8}; \frac{1}{4}\right) \cup (4; 8)$. **1064.** $(4^{\log_{0.8} 0.2}; \infty)$. **1065.** $(\sqrt[5]{5}; 5)$. **1066.** $(\log_{\sqrt{5}}(\sqrt{2}+1); \log_5 3)$. **1067.** $(0; 0.4) \cup (1; \infty)$. **1068.** $(0; 0.25) \cup (4; \infty)$. **1069.** $(1; 2) \cup (64; \infty)$. **1070.** $\left(0; \frac{1}{3}\right) \cup (243; \infty)$. **1071.** $(0; 0.5) \cup (5; \infty)$. **1072.** $(0.01; \infty)$. **1073.** $(1; 5)$. **1074.** $(0.25; 1) \cup (1; 4)$. **1075.** $(-\infty; \log_4(-1+\sqrt{3})) \cup (1.5; \infty)$. **1076.** $\left(\log_3 \frac{28}{27}; \log_3 4\right)$. **1077.** $(0; 2^{-48})$. **1078.** $(1; \infty)$. **1079.** $\left(\log_{1.5} \frac{3\sqrt{2}}{4}; 1.5\right)$. **1080.** $\left[\frac{1}{\sqrt{2}}; \frac{1}{\sqrt[5]{4}}\right) \cup (1; \sqrt{2}]$. **1081.** $[0.5; 1)$. **1082.** $(3; \infty)$. **1083.** $(0; 2) \cup (4; \infty)$. **1084.** $\left(2^{-\sqrt{2}}; \frac{1}{2}\right) \cup (1; 2^{\sqrt{2}})$. **1085.** $(-\sqrt{3}; -1.5) \cup (1.5; \sqrt{3})$. **1086.** $\left[-1; -\frac{2\sqrt{5}}{5}\right) \cup \left(\frac{2\sqrt{5}}{5}; 1\right]$. **1087.** $(-0.5; 2)$. **1088.** $\left(\frac{1}{3}; 3\right)$. **1089.** $(2^{-28}; 1)$. **1090.** $(-\infty; 0) \cup (1; \infty)$. **1091.** $(4; 10)$. **1092.** $(-\sqrt{2}; -1) \cup (1; \sqrt{2})$. **1093.** $(\log_2 \sqrt{13}; 2]$. **1094.** $(0; 4)$. **1095.** $\left(-\infty; \frac{7}{3}\right) \cup (3; \infty)$. **1096.** $(0; 0.5) \cup (2; 3)$. **1097.** $(-\infty; 0) \cup (1; 2) \cup (2; 3) \cup (4; \infty)$. **1098.** $(-3, -2) \cup (-1, 0)$. **1099.** $(5; \infty)$. **1100.** $(-2; 13)$. **1101.** $(13; 29)$. **1102.** $(40; 41) \cup (48; \infty)$. **1103.** $(-3; 2.96] \cup [22, \infty)$. **1104.** $\left(0; \frac{1}{\sqrt{6}}\right] \cup [1; \infty)$. **1105.** $(-1; -0.5) \cup (0; 1)$. **1106.** $(-\infty; 0] \cup [\log_6 5; 1)$. **1107.** $(-\infty; 0] \cup [\log_2 3; 2)$. **1108.** $[0.2; 5]$. **1109.** $(3; \infty)$. **1110.** $(-1; 0) \cup [1; \infty)$. **1111.** $(0; 1) \cup [2; \infty)$. **1112.** $(0; 1) \cup [2; \infty)$. **1113.** $(-1, 0) \cup (1.5; 2)$. **1114.** $[\log_3 0.9; 2)$. **1115.** $(0.5; 1)$. **1116.** $\left(-\frac{1}{2}; 0\right)$. **1117.** $(1; 1.5)$. **1118.** $(-1; 0) \cup (1; 3)$. **1119.** $(-\infty; \infty)$ for $a=1$; $\frac{a+3}{a-1}$ for $a \neq 1$. **1120.** $(-\infty; \infty)$ for $a=1$; $\varnothing$ for $a=2$, $a=-2$; $\frac{1}{a^2-4}$ for $\begin{cases} a \neq 1 \\ a \neq 2 \\ a \neq -2. \end{cases}$ **1121.** $\varnothing$ for $a=-3$,

$a=-1.5$, $a=0$; $-\frac{a(a^2+3a-9)}{3(2a+3)}$ for $\begin{cases} a\neq -3 \\ a\neq -1.5 \\ a\neq 0. \end{cases}$ **1122.** $\varnothing$ for $a=-2$, $a=-1$; $-2a^2-3a$ for $\begin{cases} a\neq -2 \\ a\neq -1. \end{cases}$ **1123.** $\varnothing$ for $a=-3$, $a=0$, $a=2$; $\frac{6-a}{a+3}$ for $\begin{cases} a\neq -3 \\ a\neq 0 \\ a\neq 2. \end{cases}$ **1124.** 0 for $a=0$; $x_1=a$, $x_2=3a$ for $a\neq 0$. **1125.** -2 for $a=0$; $\varnothing$ for $a<-\frac{1}{4}$; -3 for $a=-\frac{1}{4}$; $\frac{1-2a\pm\sqrt{4a+1}}{2a}$ for $-\frac{1}{4}<a<0$, $a>0$. **1126.** $-\frac{1}{5}$ for $a=\frac{1}{2}$; $\varnothing$ for $-9-\sqrt{84}<a<-9+\sqrt{84}$; $\frac{3a+1+\sqrt{a^2+18a-3}}{2a-1}$ for $a\leqslant -9-\sqrt{84}$, $-9+\sqrt{84}\leqslant a<\frac{1}{2}$, $a>\frac{1}{2}$. **1127.** $-\frac{1}{3}$ for $a=-2$; 0 for $a=1$; $x_1=\frac{1-a}{2+a}$, $x_2=\frac{1+a}{1-a}$ for $\begin{cases} a\neq -2 \\ a\neq 1. \end{cases}$ **1128.** $\varnothing$ for $6-2\sqrt{21}<a<6+2\sqrt{21}$; $\frac{-a\pm\sqrt{a^2-12a-48}}{6}$ for $a\leqslant 6-2\sqrt{21}$, $a\geqslant 6+2\sqrt{21}$. **1129.** -1 for $\begin{cases} a=-2 \\ a=1 \end{cases}$; $x_1=a+1$, $x_2=a-2$ for $\begin{cases} a\neq -2 \\ a\neq 1. \end{cases}$ **1130.** $\varnothing$ for $a=0$; a for $a\neq 0$. **1131.** $\varnothing$ for $a=0$; -1.5 for $a=1$; 1 for $a=-\frac{2}{3}$; $x_1=1$, $x_2=-\frac{a+2}{2}$ for $\begin{cases} a\neq -\frac{2}{3} \\ a\neq 0 \\ a=1. \end{cases}$ **1132.** $\varnothing$ for $a\neq 1$; $(-\infty;\ -2)\cup(-2;\ -1)\cup(-1;\ 1)\cup(1;\ 2)\cup(2;\ \infty)$ for $a=1$. **1133.** 0 for $a\geqslant -\sqrt{3}$; $x_1=0$, $x_2=\frac{1}{(a+\sqrt{3})^2}$ for $a<-\sqrt{3}$. **1134.** $\varnothing$ for $a<0$, $0<a<1$; 0 for $a=0$; $\frac{(a-1)^2}{4}$ for $a\geqslant 1$. **1135.** $\varnothing$ for $a<0$, $\frac{1}{2}\leqslant a<1$; $\frac{a^2}{2a-1}$ for $0\leqslant a<\frac{1}{2}$, $a\geqslant 1$. **1136.** $\varnothing$ for $a>-4$; $\frac{a^2+24a+16}{16}$ for $a\leqslant -4$. **1137.** $-a\sqrt{3}$ for $a<0$; $a\sqrt{3}$ for $a\geqslant 0$. **1138.** $\varnothing$ for $a<0$; $-a$ for $a\geqslant 0$. **1139.** $\varnothing$ for $a\leqslant 1$; $\frac{a}{\sqrt[3]{a^2-1}}$ for $a>1$. **1140.** $\varnothing$ for $|a|\geqslant\frac{1}{2}$; $\frac{4a^2+1}{4}$ for $|a|<\frac{1}{2}$. **1141.** $\pm\frac{a+\sqrt{a^2-16a+60}}{15-4a}$ for $a<\frac{15}{4}$; $\varnothing$ for $a\geqslant\frac{15}{4}$. **1142.** $\varnothing$ for $a\leqslant 0$, $a>3$; $1\pm\sqrt{1-2\log_9 a}$ for $0<a\leqslant 3$. **1143.** $\varnothing$ for $a>1$; 0 for $a=1$; $\pm\log_{12}(1+\sqrt{1-a})$ for $a<1$. **1144.** $\varnothing$

for $a \leqslant 3$, $a \geqslant 27$; $\log_4 \frac{a-27}{3-a}+2$ for $3<a<27$. **1145.** $\varnothing$ for $a \leqslant 0$; $2000a^{-\frac{10}{3}}$ for $a>0$. **1146.** $\varnothing$ for $a \leqslant 1$, $a>100$; $\frac{2 \pm \sqrt{4-2\log a}}{2}$ for $1<a \leqslant 100$. **1147.** $\varnothing$ for $a \leqslant 0$, $a=1$; a^6 for $\begin{cases} a>0 \\ a \neq 1. \end{cases}$ **1148.** $\varnothing$ for $a \leqslant 0$, $a=1$, $a \geqslant 2\sqrt{2}$; $4-a^2$ for $0<a<1$, $1<a<2\sqrt{2}$. **1149.** $\varnothing$ for $a \leqslant 0$, $a \geqslant 1$; $\frac{1+a^2}{1-a^2}$ for $0<a<1$. **1150.** $\varnothing$ for $a \leqslant 0$, $a=1$; $x_1=a^2$, $x_2=\frac{1}{2}$ for $0<a<1$, $a>1$. **1151.** $\varnothing$ for $a=0$; (0; 6) for $a=1$; 2 for $\begin{cases} a \neq 0 \\ a \neq 1. \end{cases}$ **1152.** $\varnothing$ for $a=0$; $x>0$ for $a=1$; 3^n, where $n=\pm 1, \pm 3, \pm 5, \ldots$, for $a=-1$; $x_1=3$, $x_2=\frac{1}{3}$ for $\begin{cases} a \neq 0 \\ a \neq -1 \\ a \neq 1. \end{cases}$ **1153.** $\varnothing$ for $a \in \boldsymbol{R}$.

1154. $\varnothing$ for $a \leqslant 0$, $a=1$, $a=2$; $a+2$ for $0<a<1$, $1<a<2$, $a=3$; $x_1=a-2$, $x_2=a+2$ for $2<a<3$, $a>3$. **1155.** $\varnothing$ for $a \leqslant 1$, $a=\sqrt{2}$; 3 for $a=2$; $x_1=a-1$, $x_2=a+1$ for $1<a<\sqrt{2}$, $\sqrt{2}<a<2$, $a>2$. **1156.** $\varnothing$ for $a=-1$; $\left(\frac{3}{a+1}; \frac{-3}{a+1}\right)$, for $a \neq -1$; **1157.** $\varnothing$ for $a=-7$; $\left(t; \frac{5-4t}{3}\right)$, where $t \in \boldsymbol{R}$, for $a=3$; $\left(\frac{5(a-3)}{a^2+4a-21}; \frac{10(a-3)}{a^2+4a-21}\right)$ for $\begin{cases} a \neq -7 \\ a \neq 3. \end{cases}$ **1158.** $\varnothing$ for $a=-1$; $(t;\ 1,\ -t)$, where $t \in \boldsymbol{R}$, for $a=1$; $\left(\frac{1+a+a^2}{a+1}; \frac{-a}{a+1}\right)$ for $\begin{cases} a \neq 1 \\ a \neq -1. \end{cases}$ **1159.** $(0;\ a)$, $(a;\ 0)$ for $a \in \boldsymbol{R}$. **1160.** $(a;\ 2a)$, $(2a;\ a)$ for $a \in \boldsymbol{R}$. **1161.** $(t_1;\ t_2;\ 1-t_1-t_2)$, where $t_1 \in \boldsymbol{R}$, $t_2 \in \boldsymbol{R}$, for $a=1$; $(-1-a;\ 1;\ a+1)$ for $a \neq 1$. **1162.** $(t;\ at;\ 2at)$, where $t \in \boldsymbol{R}$, for $a \in \boldsymbol{R}$. **1163.** $\varnothing$ for $a<0$; $(9a^2;\ a^2)$ for $a \geqslant 0$. **1164.** $\varnothing$ for $a<1$; $\left(\frac{(a+1)^2}{4}; \frac{(a-1)^2}{4}\right)$ for $a \geqslant 1$. **1165.** $\varnothing$ for $a=-1$; $(-\infty;\ 1-a)$ for $a>-1$; $(1-a;\ \infty)$ for $a<-1$. **1166.** $(-\infty;\ \infty)$ for $a=\frac{5}{9}$; $\left(-\infty; \frac{63}{9a-5}\right)$ for $a>\frac{5}{9}$; $\left(\frac{63}{9a-5}; \infty\right)$ for $a<\frac{5}{9}$. **1167.** $\varnothing$ for $a=-3$, $a=1$, $a=3$; $\left(-\infty; \frac{a+1}{a-3}\right)$ for $a<-3, 1<a<3$, $a>3$; $\left(\frac{a+1}{a-3}; \infty\right)$ for $-3<a<1$. **1168.** $(-\infty;\ 8)$ for $a=10$; $\left(\frac{3+16a}{2(10-a)}; \frac{4a}{5}\right)$ for $a>10$; $\left(-\infty; \frac{4a}{5}\right) \cup \left(\frac{3a+16}{2(10-a)}; \infty\right)$ for $a<10$. **1169.** $(-\infty;\ \infty)$ for $a=-3$; $\left(\frac{-14}{a+3}; \frac{4}{a+3}\right)$ for $a>-3$; $\left(\frac{14}{a+3}; \frac{-4}{a+3}\right)$ for $a<-3$.

1170. $\left(-\infty;\ \frac{1}{4}\right)$ for $a=-\frac{2}{5}$; $\varnothing$ for $a>\frac{2}{3}$; $-\frac{1}{2}$ for $a=\frac{2}{3}$; $(-\infty;\ \infty)$ for $a\leqslant-\frac{2}{3}$; $(-\infty;\ x_1)\cup(x_2;\ \infty)$ for $-\frac{2}{3}<a<-\frac{2}{5}$; $(x_2;\ x_1)$ for $-\frac{2}{5}<a<\frac{2}{3}$, where $x_1=\frac{-a-2+\sqrt{4-9a^2}}{5a+2}$, $x_2=\frac{-a-2-\sqrt{4-9a^2}}{5a+2}$. **1171.** $\varnothing$ for $a=-\frac{3}{2}$; $\left(\frac{1}{(2a+3)^2};\ \infty\right)$ for $a>-\frac{3}{2}$; $\left(0;\ \frac{1}{(2a+3)^2}\right)$ for $a<-\frac{3}{2}$. **1172.** $\varnothing$ for $a=2$; $\left[\frac{1}{3};\ \frac{a-1}{3}\right)$ for $a>2$; $\left(\frac{a-1}{3};\ \frac{1}{3}\right]$ for $a<2$. **1173.** $\varnothing$ for $a\leqslant 0$; $(1-2\sqrt{a};\ 1+2\sqrt{a})$ for $0<a\leqslant 1$; $[a,\ 1+2\sqrt{a})$ for $a>1$. **1174.** $[1;\infty)$ for $a\leqslant 0$; $\left[1;\ \frac{(1+a^2)^2}{4a^2}\right)$ for $0<a<1$; $\varnothing$ for $a\geqslant 1$. **1175.** $\varnothing$ for $a\leqslant -1$; $\left(\frac{a^2}{1+2a};\ -1\right]$ for $-1<a<-\frac{1}{2}$; $(-\infty;\ -1]$ for $-\frac{1}{2}\leqslant a\leqslant 0$; $(-\infty;\ -1]\cup\left[0;\ \frac{a^2}{1+2a}\right)$ for $a>0$. **1176.** $\varnothing$ for $a\leqslant 0$, $a\geqslant 4$; $(-2;\ 2)$ for $a=2$; $[-a;\ a]$ for $0<a<2$; $\left(-\frac{a}{2}\sqrt{4a-a^2};\ \frac{a}{2}\sqrt{4a-a^2}\right)$ for $2<a<4$. **1177.** $\varnothing$ for $a\leqslant -1$; $\left[-1;\ \frac{a-\sqrt{2-a^2}}{2}\right)$ for $-1<a\leqslant 1$; $\left[-1;\ \frac{a-\sqrt{2-a^2}}{2}\right]\cup\left[\frac{a+\sqrt{2-a^2}}{2};\ 1\right]$ for $1<a\leqslant\sqrt{2}$; $[-1;\ 1]$ for $a>\sqrt{2}$. **1178.** $[a;\ 0]$ for $a<0$; $\varnothing$ for $a=0$; $(0;\ a)$ for $a>0$. **1179.** $\left[a\left(1+\frac{\sqrt{2}}{2}\right);\ 0\right]$ for $a<0$; $\left[a\left(1-\frac{\sqrt{2}}{2}\right);\ 2a\right]$ for $a\geqslant 0$. **1180.** $\varnothing$ for $a\leqslant 0$, $a=1$; $\left(1;\ \frac{1+\sqrt{1+a^2}}{2}\right)$ for $0<a<1$; $\left(\frac{1+\sqrt{1+a^2}}{2};\ \infty\right)$ for $a>1$. **1181.** $(1-\sqrt{9-a};\ 1-\sqrt{1-a})\cup(1+\sqrt{1-a};\ 1+\sqrt{9-a})$ for $a\leqslant 1$; $(1-\sqrt{9-a};\ 1+\sqrt{9-a})$ for $1<a<9$; $\varnothing$ for $a\geqslant 9$. **1182.** $\left(1;\ \frac{1+\sqrt{1-4a}}{2}\right)$ for $a<0$; $\left(a;\ \frac{1-\sqrt{1-4a}}{2}\right)\cup\left(\frac{1+\sqrt{1-4a}}{2};\ 1\right)$ for $0<a\leqslant\frac{1}{4}$; $\varnothing$ for $a=0$, $a\geqslant 1$; $(a;\ 1)$ for $\frac{1}{4}<a<1$. **1183.** $\varnothing$ for $a\leqslant 0$, $a=1$; $(2;\ 3)$ for $0<a<1$, $a>1$. **1184.** $\varnothing$ for $a\leqslant 0$, $a=1$; $(a;\ 1)\cup\left(\frac{1}{a};\ \infty\right)$ for $0<a<1$; $\left(\frac{1}{a};\ 1\right)$ for $a>1$. **1185.** $\varnothing$ for $a\leqslant 1$; $(2-\sqrt{4-\log a};\ 1)\cup(2+\sqrt{4-\log a};\ \infty)$ for $1<a<1000$; $(3;\ \infty)$ for $a=1000$; $(1;\ 2-\sqrt{4-\log a})\cup(2+\sqrt{4-\log a};\ \infty)$ for $1000<a<10{,}000$; $(1;\ 2)\cup(2;\ \infty)$ for $a=10{,}000$; $(1;\ \infty)$ for $a>10{,}000$. **1186.** $\varnothing$ for $a\leqslant 0$,

$a=\frac{1}{2}$; $(1-\sqrt{1-a};\ 1+\sqrt{1-a})$ for $0<a<\frac{1}{2}$; $(1-\sqrt{1+a};\ 1-\sqrt{1-a})\cup(1+\sqrt{1-a};\ 1+\sqrt{1+a})$ for $\frac{1}{2}<a\leqslant 1$; $(1-\sqrt{1+a};\ 1+\sqrt{1+a})$ for $a>1$. **1187.** $a>\frac{11}{9}$. **1188.** $2\sqrt{2}\leqslant a\leqslant\frac{11}{3}$.
1189. $a\geqslant 2$. **1190.** $\frac{1}{2}<a<1$. **1191.** $0<a<\frac{1}{3}$. **1192.** $a<\frac{-1+\sqrt{5}}{2}$; $a>1$. **1193.** $-1<a<\frac{1+\sqrt{5}}{2}$. **1194.** $a<2$; $a>\frac{7}{3}$.
1195. $-\sqrt{2}<a<-\frac{16}{17}$; $0<a<\sqrt{2}$. **1196.** $-1<a<\frac{1-\sqrt{3}}{2}$; $1<a<\frac{1+\sqrt{3}}{2}$. **1197.** $a\leqslant -2$; $a\geqslant 0$. **1198.** $a\leqslant-\frac{1}{2}$; $a\geqslant 1$.
1199. $a=1$; $a=2$; $5\leqslant a\leqslant 6$. **1200.** $-6\leqslant a<-4$; $-3<a<-1$; $a=1$.
1201. $-6\leqslant a\leqslant -5$; $a=-2$; $a=-1$. **1202.** $-\frac{1}{2}\leqslant a\leqslant-\frac{3}{22}$; $a=1$.
1203. $2\cos\alpha$. **1204.** $-\frac{\cos 2\beta}{\sin^2\beta}$. **1205.** $\frac{1}{\cos\alpha}$. **1206.** $\tan 2\alpha$.
1207. $\sin 2\alpha$. **1208.** 1. **1209.** 1. **1210.** 1. **1211.** $\tan 3\alpha$.
1212. $\tan 4\alpha$. **1213.** $\tan n\alpha$. **1214.** $\tan\left(\frac{\alpha}{2}-\frac{\pi}{8}\right)$. **1215.** $8\cos^4\alpha$.
1216. $\tan\alpha$. **1217.** $\tan^4\alpha$. **1218.** $\frac{1}{\sqrt{2}}\sin\alpha$. **1219.** $\cos\left(\frac{\alpha}{2}+30°\right)$.
1263. $\frac{\sqrt{3}}{3}$. **1264.** $-\frac{\sqrt{2}}{2}$. **1265.** $\frac{\sqrt{6}+\sqrt{2}}{4}$. **1266.** $2-\sqrt{3}$.
1267. $\frac{-\sqrt{6}-\sqrt{2}}{4}$. **1268.** $\frac{-\sqrt{6}-\sqrt{2}}{4}$. **1269.** $\frac{\sqrt{2-\sqrt{2}}}{2}$. **1270.** 0.
1271. $\frac{3}{16}$. **1272.** 1. **1273.** 4. **1274.** $\sin\alpha=\frac{\sqrt{5}}{5}$; $\cos\alpha=-\frac{2\sqrt{5}}{5}$, $\tan\alpha=-\frac{1}{2}$. **1275.** $\sin\alpha=-\frac{4}{5}$, $\tan\alpha=\frac{4}{3}$, $\cot\alpha=\frac{3}{4}$.
1276. $\cos\alpha=\frac{5}{13}$, $\tan\alpha=-\frac{12}{5}$, $\cot\alpha=-\frac{5}{12}$. **1277.** $\sin 2\alpha=\frac{120}{169}$, $\cos 2\alpha=-\frac{119}{169}$, $\tan 2\alpha=-\frac{120}{119}$, $\cot 2\alpha=-\frac{119}{120}$. **1278.** $\frac{125}{78}$.
1279. $\frac{5-12\sqrt{3}}{26}$. **1281.** (a) m^2-2; (b) $m(m^2-3)$; (c) $\pm\sqrt{m^2-4}$.
1282. (a) $\sin\frac{\alpha}{2}=\frac{\sqrt{10}}{10}$, $\cos\frac{\alpha}{2}=\frac{3\sqrt{10}}{10}$, $\tan\frac{\alpha}{2}=\frac{1}{3}$; (b) $\sin\frac{\alpha}{2}=\frac{4}{5}$, $\cos\frac{\alpha}{2}=-\frac{3}{5}$, $\tan\frac{\alpha}{2}=-\frac{4}{3}$. **1283.** $\frac{4}{5}$. **1285.** $\frac{\sqrt{7}-2}{3}$. **1323.** $\frac{5}{12}\pi$.

1324. -1. **1325.** $-\frac{\sqrt{3}}{2}$. **1326.** $\frac{1}{2}$. **1327.** $\frac{\pi}{4}$. **1328.** 0.3π.

1329. $-\frac{\pi}{3}$. **1330.** $\frac{\pi}{4}$. **1331.** $-\frac{2\pi}{3}$. **1332.** $\frac{6}{7}\pi$. **1333.** π.

1334. $-\frac{\sqrt{3}}{3}$. **1335.** 0.2. **1336.** $\frac{\sqrt{33}}{11}$. **1337.** 0. **1338.** $\frac{13}{85}$. **1339.** $\frac{8\sqrt{5}}{81}$.

1340. $xy-\sqrt{1-x^2}\sqrt{1-y^2}$. **1341.** $xy+\sqrt{1-x^2}\sqrt{1-y^2}$. **1342.** $\frac{x+y}{1-xy}$.

1343. $\frac{x\sqrt{1-y^2}+y\sqrt{1-x^2}}{\sqrt{1-x^2}\sqrt{1-y^2}-xy}$. **1344.** $2x\sqrt{1-x^2}$. **1345.** $\frac{2x}{1-x^2}$.

1346. $\frac{1-x^2}{1+x^2}$. **1347.** $\frac{2x}{1+x^2}$. **1348.** $\frac{x^2-1}{x^2+1}$. **1349.** $\sqrt{\frac{1+x}{2}}$.

1350. $\frac{-1+\sqrt{1+x^2}}{x}$. **1432.** $\varnothing$. **1433.** $\varnothing$. **1434.** $\frac{\pi}{3}k$.

1435. $\frac{\pi}{2}k$. **1436.** $\frac{\pi}{2}+2\pi k$. **1437.** $2\pi k$. **1438.** $\frac{\pi}{6}+\frac{2}{3}\pi k$.

1439. $\pm\frac{\pi}{3}+\pi k$; $-\frac{\pi}{4}+\pi k$. **1440.** $\frac{2\pi}{3}k$; $\pm\arccos\frac{-1\pm\sqrt{5}}{4}+2\pi n$.

1441. $\frac{\pi}{4}+\frac{\pi}{2}k$; $-\frac{\pi}{2}+2\pi n$. **1442.** $(-1)^k\frac{\pi}{6}+\pi k$. **1443.** $\frac{\pi}{4}+\frac{\pi}{2}k$.

1444. πk; $(-1)^n\frac{\pi}{6}+\pi n$. **1445.** $\pm\frac{3\pi}{4}+2\pi k$. **1446.** $2\pi k$; $\frac{\pi}{6}+\frac{2\pi}{3}n$.

1447. $\frac{\pi}{12}+\frac{2}{3}\pi k$; $-\frac{\pi}{4}+2\pi n$. **1448.** $\frac{\pi}{10}+\frac{2\pi}{5}k$. **1449.** $\frac{\pi}{40}+\frac{\pi}{10}k$; $\frac{\pi}{20}+\frac{\pi}{5}n$. **1450.** $\frac{\pi}{2}+2\pi k$; $(-1)^{n+1}\frac{\pi}{6}+2\pi n$. **1451.** $\frac{\pi}{4}+\frac{\pi}{2}k$.

1452. $\pm\frac{\pi}{6}+\pi k$. **1453.** $\frac{\pi}{2}+\pi k$; $\pm\frac{\pi}{3}+2\pi n$. **1454.** $\frac{\pi}{4}+\frac{\pi}{2}k$.

1455. $\frac{\pi}{2}+2\pi k$; $(-1)^n\frac{\pi}{6}+\pi n$. **1456.** $\arctan\frac{3}{4}+\pi k$. **1457.** $\arctan\frac{2}{3}+\pi k$.

1458. $\frac{\pi}{4}+\pi k$; $-\arctan 2+\pi n$. **1459.** $\frac{\pi}{4}+\pi k$; $\arctan 3+\pi n$.

1460. $\arctan(-1\pm\sqrt{3})+\pi k$. **1461.** $\arctan\frac{3\pm\sqrt{17}}{4}+\pi k$. **1462.** $\frac{\pi}{4}k$; $\frac{\pi}{24}+\frac{\pi}{12}n$. **1463.** $\frac{\pi}{3}k$; $\frac{\pi}{2}+\pi n$. **1464.** $\pm\frac{\pi}{6}+\frac{\pi}{2}k$. **1465.** $\frac{\pi}{7}+\frac{2\pi}{7}k$; $\frac{\pi}{3}+\frac{2\pi}{3}n$. **1466.** $\frac{\pi}{2}k$; $\pm\frac{2\pi}{3}+2\pi n$. **1467.** $\frac{\pi}{2}+\pi k$; $-\frac{\pi}{8}+\frac{\pi}{2}n$.

1468. $\frac{\pi}{6}\pm\frac{\pi}{8}+\pi k$. **1469.** $\frac{\pi}{6}+\pi k$; $\frac{\pi}{12}+\frac{\pi}{4}n$. **1470.** $\frac{\pi}{3}+\frac{\pi}{2}k$.

1471. $-\frac{\pi}{32}+\frac{\pi}{4}k$; $\frac{\pi}{72}+\frac{\pi}{9}n$. **1472.** $\frac{\pi}{2}+\pi k$; $\arctan\frac{3}{2}+\pi n$.

1473. πk; $\frac{\pi}{4}+\pi n$. **1474.** $\frac{\pi}{2}+\pi k$; $\arctan 7+\pi n$; $\arctan 3+\pi m$.

1475. $-\arctan\frac{4}{3}+\pi k$. **1476.** $\frac{\pi}{2}k$. **1477.** $\varnothing$ **1478.** $\varnothing$. **1479.** $\varnothing$. **1480.** $\varnothing$. **1481.** $(-1)^k\frac{\pi}{6}+\pi k$. **1482.** $\pi+2\pi k$; $2\arctan\frac{3}{2}+2\pi n$. **1483.** $\arctan\frac{3\pm\sqrt{6}}{3}+\pi n$. **1484.** $\frac{\pi}{2}k$; $\frac{1}{2}\arctan 2+\pi n$. **1485.** $\frac{\pi}{4}+\pi k$. **1486.** $2\pi k$; $2\arctan(-2)+2\pi n$. **1487.** $\frac{\pi}{2}+2\pi k$; $2\pi n$. **1488.** $\varnothing$. **1489.** $\frac{\pi}{2}+2\pi k$; $\pi+2\pi n$. **1490.** $2\pi k$; $-\frac{\pi}{2}+2\pi n$. **1491.** $\frac{\pi}{4}\pm\arccos\frac{\sqrt{2}}{10}+2\pi k$. **1492.** $\pm\frac{2\pi}{3}+2\pi k$. **1493.** $\frac{2\pi k}{15}\left(k\neq\frac{15l}{2}\right)$; $\frac{\pi}{17}+\frac{2\pi}{17}n$ $(n\neq 17m+8)$. **1494.** $-\frac{\pi}{66}+\frac{\pi}{11}k$; $\frac{\pi}{9}+\frac{\pi}{6}n$. **1495.** $\pi+2\pi k$; $(-1)^n\frac{\pi}{2}+2\pi n$. **1496.** $2\pi+4\pi k$; $(-1)^n\frac{2\pi}{3}+4\pi n$. **1497.** $\frac{\pi}{4}+\pi k$. **1498.** $\frac{7\pi}{12}+\pi k$. **1499.** $(-1)^k\frac{\pi}{6}+\pi k$. **1500.** $3\pi k$; $\pm\frac{\pi}{4}+\frac{3\pi}{2}n$. **1501.** $-\frac{\pi}{2}+\pi k$; $\frac{\pi}{4}\pm\arccos\frac{1}{2\sqrt{2}}+2\pi n$. **1502.** $\frac{\pi}{2}+\pi k$; $\pm\frac{1}{2}\arccos\left(-\frac{1}{4}\right)+\pi n$. **1503.** $\frac{\pi}{4}+\pi k$; $(-1)^n\frac{1}{2}\arcsin\frac{\sqrt{5}-1}{2}+\frac{\pi}{2}n$. **1504.** $-\frac{\pi}{4}+\pi k$; $\frac{\pi}{4}\pm\arccos\frac{\sqrt{2}-\sqrt{10}}{4}+2\pi n$. **1505.** $\frac{\pi}{4}+\pi k$. **1506.** $\pm\frac{\pi}{12}+\frac{\pi}{4}k$. **1507.** $\frac{\pi}{8}+\frac{\pi}{4}k$. **1508.** $\frac{\pi}{8}+\frac{\pi}{4}k$. **1509.** $2\pi k$; $\frac{5\pi}{4}+2\pi n$. **1510.** $\frac{2\pi}{3}+2\pi k$. **1511.** $(-1)^k\frac{\pi}{6}+\pi k$. **1512.** $\pm\frac{\pi}{3}+2\pi k$. **1513.** $\frac{\pi}{4}+\pi k$. **1514.** $\varnothing$. **1515.** $-\arctan\frac{1}{6}+\pi k$; $-\arctan\frac{1}{3}+\pi n$. **1516.** $\frac{\pi}{2}+2\pi k$. **1517.** $\pm\frac{\pi}{4}+2\pi k$. **1518.** $-\frac{3\pi}{8}+\pi k$. **1519.** $\frac{\pi}{4}+2\pi k$; $-\arctan 3+\pi(2n+1)$. **1520.** $\frac{5\pi}{4}+2\pi k$; $\arctan 3+\pi(2n+1)$. **1521.** $-\frac{\pi}{4}+\pi k$; $2\pi n$. **1522.** $\varnothing$. **1523.** $\pm\frac{1}{2}\arccos\frac{1}{3}+\pi k$; $\pm\frac{1}{2}\arccos\left(-\frac{1}{4}\right)+\pi n$. **1524.** $\varnothing$. **1525.** $\frac{\pi}{2}+\pi k$; $\frac{\pi}{8}+\frac{\pi}{4}n$. **1526.** $\pi+4\pi k$. **1527.** $\pi+2\pi k$. **1528.** $\frac{\pi}{2}+2\pi k$. **1529.** $\frac{3\pi}{2}+3\pi k$. **1530.** $\frac{\pi}{6}+\pi k$. **1531.** $\varnothing$. **1532.** $2\pi+24\pi k$. **1533.** $\varnothing$. **1534.** πk. **1535.** 0. **1536.** $\frac{\pi}{4}+2\pi k$.

1537. πk. 1538. $\pm\frac{1}{6}+k\ (k\in \boldsymbol{Z})$. 1539. $\frac{\pi}{2}+\pi k$; $\pm \arctan\frac{1}{2}+\pi n$. 1540. 1; 2. 1541. $\frac{1+\sqrt{2}}{3}$. 1542. $\sin\frac{1}{2}$. 1543. $\tan\frac{2}{5}\pi$. 1544. 0. 1545. $\frac{-1+\sqrt{3}}{2}$. 1546. 0. 1547. $\sqrt{\frac{\sqrt{5}-1}{2}}$. 1548. $\frac{2}{3}$. 1549. $\frac{1}{2}$. 1550. 4. 1551. $\left(-\frac{\pi}{2}+2\pi k;\ \frac{\pi}{2}+2\pi n\right)$. 1552. $(k;\ 0)$, $(k;\ 2)$, $k\in \boldsymbol{Z}$. 1553. $\left(\frac{\pi}{4}+2\pi k;\ \frac{\pi}{2}+2\pi n\right)$. 1554. $\left(-1;\ 1+\frac{\pi}{4}(2n+1)\right)$. 1555. $\left(-\frac{\pi}{6}+\frac{\pi}{2}k;\ \frac{\pi}{6}+\frac{\pi}{4}n\right)$. 1556. $\left(1;\ \frac{\pi}{2}+\pi k\right)$, $\left(-1;\ \frac{\pi}{2}+\pi n\right)$. 1557. $\left(\frac{\pi}{2}+\pi k;\ \frac{\pi}{2}+\pi n\right)$. 1558. $\left(\pi k;\ \frac{\pi}{2}n;\ -\frac{\pi}{6}+\frac{2\pi}{3}m\right)$. 1559. (2; 7), (4; -13), (1; 2), (5; -8), (8; -5), (13; -4). 1560. $\left(\frac{\pi}{2}(k+n);\ \frac{\pi}{2}(k-n)\right)$. 1561. $\left(\frac{\pi}{6}+\pi(k-n);\ \frac{\pi}{3}+\pi(k+n)\right)$, $\left(-\frac{\pi}{6}+\pi(k-n);\ \frac{2\pi}{3}+\pi(k+n)\right)$. 1562. $\left((-1)^k\frac{\pi}{6}+\pi k;\ \pm\frac{2\pi}{3}+2\pi n\right)$, $\left((-1)^{k+1}\frac{\pi}{6}+\pi k;\ \pm\frac{\pi}{3}+2\pi n\right)$. 1563. $\left(\frac{\pi}{6}+\pi k;\ \frac{\pi}{6}-\pi n\right)$. 1564. $\left((-1)^k\arcsin\left(1-\frac{\sqrt{2}}{2}\right)+\pi k;\ \arccos(2-\sqrt{2})+2\pi n\right)$, $\left((-1)^k\arcsin\left(1-\frac{\sqrt{2}}{2}\right)+\pi k;\ -\arccos(2-\sqrt{2})+2\pi n\right)$. 1565. $\left(\frac{\pi}{2}+\pi k;\ \frac{\pi}{3}+2\pi n\right)$, $\left(\frac{\pi}{2}+\pi k;\ -\frac{\pi}{3}+2\pi n\right)$, $\left(\frac{\pi}{3}+2\pi k;\ \frac{\pi}{2}+\pi n\right)$, $\left(-\frac{\pi}{3}+2\pi k;\ \frac{\pi}{2}+\pi n\right)$. 1566. $(t;\ t-\pi(2n+1))$, where $t\in \boldsymbol{R}$. 1567. $\left(\frac{1}{6}+k;\ -\frac{1}{6}+k\right)$, where $k\in \boldsymbol{Z}$. 1568. $\left(\frac{\pi}{3}+\pi k;\ -\pi k\right)$, $\left(\pi k;\ \frac{\pi}{3}-\pi k\right)$. 1569. $\left(\frac{\pi}{2}+\pi k;\ \frac{\pi}{3}-\pi k\right)$, $\left(\frac{\pi}{3}+\pi k;\ \frac{\pi}{2}-\pi k\right)$. 1570. $\left(\pi k;\ \frac{\pi}{4}-\pi k\right)$, $\left(\frac{3\pi}{4}+\pi k;\ -\frac{\pi}{2}-\pi k\right)$. 1571. $\left(\frac{\pi}{2}+4\pi k;\ 2\pi+4\pi n\right)$, $\left(-\frac{\pi}{2}+4\pi k;\ 2\pi+4\pi n\right)$, $\left(2\pi+4\pi k;\ \frac{\pi}{2}+4\pi n\right)$ $\left(2\pi+4\pi k;\ -\frac{\pi}{2}+4\pi n\right)$. 1572. $\left(\pi k;\ \frac{3\pi}{4}-\pi k\right)$, $\left(\frac{\pi}{4}+\pi k;\ \frac{\pi}{2}-\pi k\right)$. 1573. $\left(\frac{5\pi}{24}+\frac{\pi}{2}k;\ \frac{\pi}{24}+\frac{\pi}{2}k\right)$. 1574. $\varnothing$. 1575. $\left(\frac{\pi}{4}+2\pi k;\ \frac{\pi}{6}+2\pi n\right)$. 1576. $\left(\frac{\pi}{6}+\pi(n+k);\ \frac{\pi}{6}+\pi(n-k)\right)$, $\left(-\frac{\pi}{6}+\pi(n+k);\ -\frac{\pi}{6}+\pi(n-k)\right)$. 1577. $(\alpha+\pi(k+n);\ -\beta+\pi(k-n))$, $\left(\frac{\pi}{2}-\beta+\pi(k+n)\right);\ -\frac{\pi}{2}+\alpha+\pi(k-n)\Big)$, $\left(\frac{\pi}{2}+\beta+\pi(k+n);\right.$

$\frac{\pi}{2} - \alpha + \pi(k-n)\Big)$, $(\pi - \alpha + \pi(k+n);\ \beta - \pi(k-n))$, where $\alpha = \frac{1}{2}\left(\arcsin\frac{2}{5} + \arcsin\frac{4}{5}\right)$, $\beta = \frac{1}{2}\left(\arcsin\frac{4}{5} - \arcsin\frac{2}{5}\right)$.

1578. $\left(\pm\frac{\pi}{3} + 2\pi k;\ \pm\frac{\pi}{3} + 2\pi n\right)$. **1579.** $(\alpha + \pi(n+k);\ \beta + \pi(n-k))$, $(\beta + \pi(n+k);\ \alpha + \pi(n-k))$, $(-\beta + \pi(n+k);\ -\alpha + \pi(n-k))$, $(-\alpha + \pi\times(n+k);\ -\beta + \pi(n-k))$, where $\alpha = \frac{1}{2}\arccos\frac{\sqrt{2}}{4} + \frac{\pi}{8}$, $\beta = \frac{1}{2}\arccos\frac{\sqrt{2}}{4} - \frac{\pi}{8}$.

1580. $(\pi k;\ \pi n)$, $(\alpha + \pi k;\ \beta + \pi n)$, $(-\alpha + \pi k;\ -\beta + \pi n)$, where $\alpha = \arcsin\frac{\sqrt{6}}{3}$, $\beta = \arcsin\frac{\sqrt{6}}{9}$, k and n are either both even or both odd.

1581. $\left(\alpha + \pi k;\ \frac{\pi}{4} - \alpha - \pi k\right)$, $\left(\beta + \pi k;\ \frac{\pi}{4} - \beta - \pi k\right)$, where $\alpha = \arctan\frac{-7+\sqrt{97}}{8}$, $\beta = -\arctan\frac{7+\sqrt{97}}{8}$. **1582.** $\left(\frac{\pi}{4} + \pi k;\ -\frac{\pi}{12} - \pi k\right)$, $\left(-\frac{\pi}{12} + \pi k;\ \frac{\pi}{4} - \pi k\right)$. **1583.** $\left(\frac{3\pi}{2} + \pi k;\ -\frac{\pi}{6} + \pi k\right)$. **1584.** $\left(\arctan\frac{1}{2} + \pi k;\ \frac{\pi}{4} - \arctan\frac{1}{2} - \pi k\right)$, $\left(\arctan\frac{1}{3} + \pi k;\ \frac{\pi}{4} - \arctan\frac{1}{3} - \pi k\right)$, $\left(\frac{\pi}{4} - \arctan\frac{1}{2} - \pi k;\ \arctan\frac{1}{2} + \pi k\right)$, $\left(\frac{\pi}{4} - \arctan\frac{1}{3} - \pi k;\ \arctan\frac{1}{3} + \pi k\right)$.

1585. $\left(\frac{\pi}{6} + 2\pi k;\ \frac{\pi}{4} + 2\pi n\right)$, $\left(\frac{5\pi}{6} + 2\pi k;\ \frac{3\pi}{4} + 2\pi n\right)$, $\left(-\frac{\pi}{6} + 2\pi k;\ -\frac{\pi}{4} + 2\pi n\right)$, $\left(-\frac{5\pi}{6} + 2\pi k;\ -\frac{3\pi}{4} + 2\pi n\right)$. **1586.** $\left((-1)^k\frac{\pi}{6} + \pi k;\ (-1)^{n+1}\frac{\pi}{6} + \arctan\frac{5}{2} + \pi(k-n)\right)$. **1587.** $\left(\frac{\pi}{2}(k-n);\ \frac{\pi}{2}n\right)$, $\left(-\frac{\pi}{6} + \pi(k-2n);\ \frac{\pi}{3} + \pi n\right)$, $\left(\frac{7\pi}{6} + \pi(k-2n);\ -\frac{\pi}{3} + \pi n\right)$. **1588.** $\left(\frac{\pi}{2}k;\ \pi(n+k)\right)$.

1589. $\left(-\frac{\pi}{6} + (-1)^k\alpha + \frac{\pi}{2}k;\ \frac{\pi}{6} + (-1)^k\alpha + \frac{\pi k}{2}\right)$, where $\alpha = \frac{1}{2}\arcsin\frac{2-3\sqrt{3}}{6}$.

1590. $\left(2\pi k;\ \frac{\pi}{2} + 2\pi n\right)$, $\left(\pi + 2\pi k;\ -\frac{\pi}{2} + 2\pi n\right)$, $\left(\frac{\pi}{3} + 2\pi k;\ \frac{\pi}{6} + 2\pi n\right)$, $\left(-\frac{\pi}{3} + 2\pi k;\ \frac{5\pi}{6} + 2\pi n\right)$, $\left(\frac{2\pi}{3} + 2\pi k;\ \frac{7\pi}{6} + 2\pi n\right)$, $\left(-\frac{2\pi}{3} + 2\pi k;\ -\frac{\pi}{6} + 2\pi n\right)$. **1591.** $\left(\frac{\pi}{2} + \pi k;\ \frac{\pi}{6} - \pi k\right)$. **1592.** $\left(\frac{\pi}{2} + 2\pi k;\ \frac{\pi}{6} + 2\pi n\right)$, $\left(-\frac{\pi}{6} + 2\pi k;\ -\frac{\pi}{2} + 2\pi n\right)$. **1593.** $(2\pi k;\ \pi + 2\pi n)$.

1594. $\left(\frac{7\pi}{24} + \pi(k+n);\ \frac{\pi}{24} + \pi(k-n)\right)$, $\left(\frac{\pi}{24} + \pi(k+n);\ \frac{7\pi}{24} + \pi(k-n)\right)$, $\left(-\frac{\pi}{24} + \pi(k+n);\ -\frac{7\pi}{24} + \pi(k-n)\right)$,

$\left(-\frac{7\pi}{24}+\pi(k+n);\ -\frac{\pi}{24}+\pi(k-n\right)$. **1595.** $\left(\frac{\pi}{4}+\frac{\pi}{2}(k+n);\ \frac{\pi}{2}+\pi(k-n)\right)$.

1596. $\left(\frac{\pi}{6}+\pi k;\ -\frac{\pi}{6}+\pi n\right)$, $\left(-\frac{\pi}{6}+\pi k;\ \frac{\pi}{6}+\pi n\right)$. **1597.** $\left(\frac{\pi}{6}+\frac{\pi}{2}k;\ -\frac{1}{2}\arctan\frac{1}{2}+\frac{\pi}{2}n\right)$, $\left(-\frac{\pi}{6}+\frac{\pi}{2}k;\ -\frac{1}{2}\arctan\frac{1}{2}+\frac{\pi}{2}n\right)$. **1598.** $(\pi k;\ 2\pi n)$, $\left(\frac{\pi}{2}+\pi k;\ \frac{\pi}{2}+2\pi n\right)$. **1599.** $\left(\frac{\pi}{4}+\pi k;\ \arctan 2+\pi n;\ \frac{3\pi}{4}-\arctan 2-\pi(k+n)\right)$, $\left(-\frac{\pi}{4}+\pi k;\ -\arctan 2+\pi n;\ \frac{5\pi}{4}+\arctan 2-\pi(k+n)\right)$.

1600. $\left(\frac{\pi}{4}+\pi k;\ \arctan 2+\pi n;\ \frac{3\pi}{4}-\arctan 2-\pi(k+n)\right)$, $\left(\arctan 2+\pi n;\ \frac{\pi}{4}+\pi k;\ \frac{3\pi}{4}-\arctan 2-\pi(k+n)\right)$. **1601.** $\left((\pi k;\ \pi n;\ \pi-\pi(k+n)\right)$, $\left(\frac{\pi}{2}+2\pi k;\ \frac{\pi}{6}+2\pi n;\ \frac{\pi}{3}-2\pi(k+n)\right)$, $\left(-\frac{\pi}{2}+2\pi k;\ -\frac{\pi}{6}+2\pi n;\ \frac{5\pi}{6}-2\pi(k+n)\right)$. **1602.** $\left(\frac{\pi}{6}+\pi k;\ \frac{\pi}{6}+\pi n;\ \frac{\pi}{4}+\frac{\pi}{2}m\right)$, $\left(-\frac{\pi}{6}+\pi k;\ \frac{\pi}{6}+\pi n;\ \frac{\pi}{4}+\frac{\pi}{2}m\right)$, $\left(\frac{\pi}{6}+\pi k;\ -\frac{\pi}{6}+\pi n;\ \frac{\pi}{4}+\frac{\pi}{2}m\right)$, $\left(-\frac{\pi}{6}+\pi k;\ -\frac{\pi}{6}+\pi n;\ \frac{\pi}{4}+\frac{\pi}{2}m\right)$. **1603.** $\left(\frac{5\pi}{6};\ \frac{7\pi}{6}\right)$, $\left(\frac{\pi}{6};\ \frac{11\pi}{6}\right)$, $\left(\frac{7\pi}{6};\ \frac{7\pi}{6}\right)$, $\left(\frac{11\pi}{6};\ \frac{11\pi}{6}\right)$. **1604.** (1) $-\frac{\pi}{6}+2\pi k<x<\frac{7\pi}{6}+2\pi k$; (2) $\frac{\pi}{6}+2\pi k<x<\frac{11\pi}{6}+2\pi k$; (3) $-\frac{\pi}{6}+\pi k\leqslant x<\frac{\pi}{2}+\pi k$; (4) $\frac{3\pi}{4}+\pi k\leqslant x<\pi+\pi k$.

1605. (1) $\pi-\arcsin\frac{1}{5}+2\pi k<x<2\pi+\arcsin\frac{1}{5}+2\pi k$; (2) $-\arccos(-0.7)+2\pi k\leqslant x\leqslant\arccos(-0.7)+2\pi k$; (3) $-\frac{\pi}{2}+\pi k<x\leqslant\arctan 5+\pi k$; (4) $\pi k<x<\operatorname{arccot}\left(-\frac{\sqrt{3}}{4}\right)+\pi k$. **1606.** $\frac{5\pi}{6}+2\pi k<x<\frac{5\pi}{3}+2\pi k$. **1607.** $-\frac{\pi}{3}+2\pi k<x\leqslant 2\pi k$; $\frac{\pi}{2}+2\pi k<x\leqslant\pi+2\pi k$. **1608.** $\frac{\pi}{4}+2\pi k\leqslant x<\frac{5\pi}{6}+2\pi k$; $\pi+2\pi k<x\leqslant\frac{7\pi}{4}+2\pi k$.

1609. $\pi k<x<\frac{\pi}{4}+\pi k$; $\frac{\pi}{2}+\pi k<x\leqslant\frac{2\pi}{3}+\pi k$. **1610.** $\arccos\frac{1}{5}+2\pi k<x<\pi-\arcsin\frac{1}{5}+2\pi k$. **1611.** $-\frac{\pi}{2}+2\pi k<x<\arctan 3+2\pi k$; $\frac{\pi}{2}+2\pi k<x\leqslant\arccos\left(-\frac{3}{5}\right)+2\pi k$; $\pi+\arccos\frac{3}{5}+2\pi k\leqslant x<\pi+\arctan 3+2\pi k$.

1612. $\operatorname{arccot} 2+2\pi k<x<\arcsin\frac{4}{7}+2\pi k$; $\pi-\arcsin\frac{4}{7}+2\pi k<x<\pi+2\pi k$; $\pi+\operatorname{arccot} 2+2\pi k<x<2\pi+2\pi k$. **1613.** $\operatorname{arccot} 0.3+\pi k\leqslant x<\frac{\pi}{2}+\pi k$.

1614. $\arcsin\frac{2}{3}+2\pi k<x<\frac{3\pi}{2}+2\pi k$. **1615.** $-\infty<x<\infty$. **1616.** $\operatorname{arccot} 7+\pi k\leqslant x<\pi+\pi k$. **1617.** $\operatorname{arccot}\sqrt{2}+\pi k<x<\frac{7\pi}{6}+\pi k$. **1618.** $-\sqrt{\frac{\pi}{3}}\leqslant x\leqslant\sqrt{\frac{\pi}{3}}$; $\sqrt{2\pi k-\frac{\pi}{3}}\leqslant x\leqslant\sqrt{\frac{\pi}{3}+2\pi k}$; $-\sqrt{\frac{\pi}{3}+2\pi k}\leqslant x\leqslant-\sqrt{-\frac{\pi}{3}+2\pi k}$, where $k\in N$. **1619.** $\frac{\pi}{3}+\pi k<x<\pi+\pi k$. **1620.** $\frac{13\pi}{36}+\frac{2\pi}{3}k<x<\frac{19\pi}{36}+\frac{2\pi}{3}k$. **1621.** $-\frac{\pi}{3}+2\pi k<x<\frac{\pi}{3}+2\pi k$. **1622.** $\frac{\pi}{2}+2\pi k<x<\frac{3\pi}{2}+2\pi k$. **1623.** $\frac{\pi}{12}+\frac{2\pi}{3}k<x<\frac{5\pi}{12}+\frac{2\pi}{3}k$. **1624.** $\pi k<x<\frac{\pi}{4}+\pi k$; $\arctan 3+\pi k<\frac{\pi}{2}+\pi k$. **1625.** $\frac{\pi}{6}+2\pi k<x<\frac{\pi}{2}+2\pi k$; $\frac{\pi}{2}+2\pi k<x<\frac{5\pi}{6}+2\pi k$. **1626.** $-\frac{\pi}{2}+2\pi k<x<-\frac{\pi}{3}+2\pi k$; $\frac{\pi}{3}+2\pi k<x<\frac{\pi}{2}+2\pi k$. **1627.** $-\frac{\pi}{3}+\pi k<x<\frac{\pi}{4}+\pi k$; $\frac{\pi}{3}+\pi k<x<\frac{\pi}{2}+\pi k$. **1628.** $-\frac{\pi}{12}+\frac{\pi}{3}k<x<\frac{\pi}{12}+\frac{\pi}{3}k$. **1629.** $\operatorname{arccot}\frac{1}{3}+\pi k<x<\operatorname{arccot}\left(-\frac{5}{12}\right)+\pi k$. **1630.** $-\infty<x<\infty$. **1631.** $-\frac{\pi}{3}+\pi k<x<\pi k$. **1632.** $\frac{\pi}{4}+\pi k<x<\operatorname{arccot}\left(-\frac{1}{3}\right)+\pi k$. **1633.** $\frac{\pi}{6}+\pi k<x<\frac{\pi}{3}+\pi k$. **1634.** $\frac{\pi}{2}k<x<\frac{\pi}{8}+\frac{\pi}{2}k$. **1635.** $-\frac{\pi}{4}+\frac{\pi}{2}k<x<-\frac{\pi}{8}+\frac{\pi}{2}k$. **1636.** $-\frac{\pi}{4}+2\pi k<x<\frac{\pi}{2}+2\pi k$; $\frac{\pi}{2}+2\pi k<x<\frac{5\pi}{4}+2\pi k$. **1637.** $-\frac{\pi}{2}+2\pi k<x<2\pi k$; $\frac{\pi}{4}+2\pi k<x<\frac{\pi}{2}+2\pi k$; $\pi+2\pi k<x<\frac{5\pi}{4}+2\pi k$. **1638.** $\frac{\pi}{6}+\frac{\pi}{2}k<x<\frac{\pi}{3}+\frac{\pi}{2}k$. **1639.** $-\frac{\pi}{3}+2\pi k\leqslant x<2\pi k$; $\frac{\pi}{3}+2\pi k\leqslant x<\pi+2\pi k$. **1640.** $\frac{\pi}{6}+\pi k\leqslant x\leqslant\frac{5\pi}{6}+\pi k$. **1641.** $2\operatorname{arccot} 2+2\pi k<x<2\operatorname{arccot}\left(-\frac{1}{2}\right)+2\pi k$. **1642.** $\frac{\pi}{12}+2\pi k<x<\frac{3\pi}{4}+2\pi k$; $\frac{17\pi}{12}+2\pi k<x<\frac{7\pi}{4}+2\pi k$. **1643.** $-\frac{1}{2}\arccos\frac{1}{3}+\pi k<x<-\frac{\pi}{6}+\pi k$; $\pi k<x<\frac{\pi}{6}+\pi k$; $\frac{1}{2}\arccos\frac{1}{3}+\pi k<x<\frac{\pi}{4}+\pi k$; $\frac{2\pi}{3}+\pi k<x<\frac{3\pi}{4}+\pi k$; $\frac{\pi}{3}+\pi k<x<\frac{\pi}{2}+\pi k$. **1644.** $-\frac{\pi}{5}+2\pi k<x<2\pi k$; $2\pi k<x<\frac{\pi}{5}+2\pi k$; $\frac{2\pi}{5}+2\pi k<x<\frac{\pi}{2}+2\pi k$; $\frac{3\pi}{5}+2\pi k<x<\frac{4\pi}{5}+2\pi k$; $\frac{6\pi}{5}+2\pi k<x<\frac{7\pi}{5}+2\pi k$; $\frac{3\pi}{2}+2\pi k<$

$x<\frac{8\pi}{5}+2\pi k$. **1645.** $-\frac{\pi}{10}+\frac{2\pi}{5}k<x<-\frac{\pi}{30}+\frac{2\pi}{5}k$; $\frac{\pi}{10}+\frac{2\pi}{5}k<x<\frac{7\pi}{30}+\frac{2\pi}{5}k$. **1646.** $-\frac{\pi}{3}+\pi k<x<-\frac{\pi}{9}+\pi k$; $\pi k<x<\frac{2\pi}{9}+\pi k$; $\frac{\pi}{2}+\pi k<x<\frac{5\pi}{9}+\pi k$. **1647.** $-\frac{\pi}{4}+2\pi k<x<\frac{\pi}{6}+2\pi k$; $\frac{\pi}{4}+2\pi k<x<\frac{3\pi}{4}+2\pi k$; $\frac{5\pi}{6}+2\pi k<x<\frac{5\pi}{4}+2\pi k$. **1648.** $-\frac{\pi}{8}+\pi k<x<\pi k$; $\frac{\pi}{2}+\pi k<x<\frac{5\pi}{8}+\pi k$; $\frac{\pi}{8}+\pi k<x<\frac{3\pi}{8}+\pi k$. **1649.** $-\frac{7\pi}{12}+\pi k\leqslant x<-\frac{\pi}{2}+\pi k$; $-\frac{\pi}{2}+\pi k<x\leqslant\frac{\pi}{12}+\pi k$. **1650.** $-\frac{7\pi}{18}+\frac{2\pi}{3}k<x<\frac{\pi}{18}+\frac{2\pi}{3}k$. **1651.** $-\frac{\pi}{3}+2\pi k<x<\frac{\pi}{3}+2\pi k$. **1652.** $\frac{\pi}{2}+10\pi k<x<\frac{3\pi}{2}+10\pi k$; $\frac{10\pi}{3}+10\pi k<x<\frac{7\pi}{2}+10\pi k$; $\frac{9\pi}{2}+10\pi k<x<5\pi+10\pi k$; $\frac{20\pi}{3}+10\pi k<x<\frac{25\pi}{3}+10\pi k$. **1653.** $-\frac{\pi}{3}+2\pi k<x<\frac{\pi}{3}+2\pi k$; $\frac{8\pi}{3}+4\pi k<x<\frac{10\pi}{3}+4\pi k$. **1654.** πk for $a<-2$, $a>2$; $x_1=\pi k$, $x_2=\frac{1}{3}(-1)^k\arcsin\frac{a}{2}+\frac{\pi}{3}k$ for $-2\leqslant a\leqslant 2$. **1655.** $\varnothing$ for $a<-8$, $a>8$; $\frac{\pi}{6}+(-1)^k\arcsin\frac{a}{8}+\pi k$ for $-8\leqslant a\leqslant 8$. **1656.** $x\in\boldsymbol{R}$ for $a=2\pi k$; $x_1=\pi+2\pi k$, $x_2=a+\pi+2\pi k$ for $a\neq 2\pi k$. **1657.** $x\in\boldsymbol{R}$ for $a=\pi+2\pi k$; $-\frac{a}{2}(-1)^k\frac{\pi}{6}+\pi k$ for $a\neq\pi+2\pi k$. **1658.** $x\in\boldsymbol{R}$ for $\begin{cases}a=0\\ b=0;\end{cases}$ $x_1=\frac{\pi}{2}+\pi k$, $x_2=-\frac{\pi}{4}+\pi k$ for $a=-b\neq 0$; $x_1=-\frac{\pi}{4}+\pi k$, $x_2=\arctan\frac{b-a}{b+a}+\pi k$ for $\begin{cases}a\neq -b\\ b\neq 0;\end{cases}$ $-\frac{\pi}{4}+\pi k$ for $\begin{cases}a\neq 0\\ b=0.\end{cases}$ **1659.** $\arcsin\frac{1-a}{\sqrt{2a^2+2}}+(-1)^k\arcsin\frac{a\sqrt{2}}{a^2+1}+\pi k$ for $a\in\boldsymbol{R}$. **1660.** $x\in\boldsymbol{R}$ for $a=\pi k$; $\frac{\pi}{4}+\pi k$ for $a\neq\pi k$. **1661.** $2\pi k$ for a rational a; 0 for an irrational a. **1662.** $\varnothing$ for $a<\frac{1}{4}$, $a>1$; $\pm\frac{1}{4}\arccos\frac{8a-5}{8}+\frac{\pi}{2}k$ for $\frac{1}{4}\leqslant a\leqslant 1$. **1663.** $\varnothing$ for $a<-2$, $a>2$; $\frac{\pi}{4}+\pi k$ for $a=-2$; $\frac{1}{2}(-1)^k\arcsin\left(1-\sqrt{a+2}\right)+\frac{\pi}{2}k$. **1664.** $\varnothing$ for $a=\frac{\pi}{2}+\pi k$; $\varnothing$ for $a=\frac{\pi}{4}+\pi n$; $-\frac{\pi}{4}-a+\pi m$ for $\begin{cases}a\neq\frac{\pi}{2}+\pi k\\ a\neq\frac{\pi}{4}+\pi n.\end{cases}$ **1665.** $x\in\boldsymbol{R}$ for $\begin{cases}a\in\boldsymbol{R}\\ b=0;\end{cases}$ $x_1=\frac{\pi}{2}+2\pi k$,

$x_2=2\pi n$ for $\begin{cases} a\in \mathbf{R} \\ b\neq 0.\end{cases}$ **1666.** $\varnothing$ for $\begin{cases} a\in \mathbf{R} \\ b=0;\end{cases}$ $x\neq \frac{\pi}{b}k$ for $\begin{cases} a=0 \\ b\neq 0;\end{cases}$ $x=\frac{\pi}{a}k$ $\left(\text{where } k\in \mathbf{Z}, \text{ but } k\neq \frac{na}{b}\right)$ for $\begin{cases} a\neq 0 \\ b\in\mathbf{R}.\end{cases}$ **1667.** πk for $a<-1$, $a\geqslant 3$; $x_1=\pi k$, $x_2=\pm\frac{1}{2}\arccos\frac{a-1}{2}+\pi k$ for $-1\leqslant a<3$. **1668.** $\frac{\pi}{2}+\pi k$ for $a\leqslant -3$, $a>1$; $x_1=\frac{\pi}{2}+\pi k$, $x_2=\pm\frac{1}{2}\arccos\frac{a+1}{2}+\pi k$ for $-3<a\leqslant 1$. **1669.** $\varnothing$ for $a=\frac{\pi}{2}(2k-4n-1)$; $-\frac{a}{2}+\frac{\pi}{4}(2k+1)$ for $a\neq\frac{\pi}{2}(2k-4n-1)$. **1670.** $\frac{\pi}{2}+\pi k$ for $a=\frac{\pi}{2}+\pi n$; $x_1=\frac{\pi}{2}+\pi k$, $x_2=-a+\frac{\pi}{2}+\pi k$ for $a\neq\frac{\pi}{2}+\pi n$. **1671.** $x_1=\frac{\pi+2a}{14}+\frac{2\pi}{7}k$, $x_2=\frac{\pi+2a}{10}+\frac{2\pi}{5}k$ for $a\in\mathbf{R}$. **1672.** $\varnothing$ for $a\neq 1$; $x\in\mathbf{Z}$ for $a=1$. **1673.** $\varnothing$ for $a\neq\frac{4n}{4k+1}$; $\frac{\pi}{2}+2\pi k$ for $a=\frac{4n}{4k+1}$. **1674.** $\varnothing$ for $a<-5$, >3; $(-1)^k\arcsin\left(-2+\sqrt{4-a}\right)+\pi k$ for $-5\leqslant a\leqslant 3$. **1675.** $\varnothing$ for $a<-4$, $a>2$; $\pm\arccos\frac{3-\sqrt{9-4a}}{2}+2\pi k$ for $-4\leqslant a\leqslant 2$. **1676.** $\varnothing$ for $a<-\sqrt{2}$, $a>\sqrt{2}$; $\pm\frac{1}{2}\arccos\left(3-2\sqrt{3-a^2}\right)+\pi k$ for $-\sqrt{2}\leqslant a\leqslant\sqrt{2}$. **1677.** $\frac{\pi}{4}+2\pi k$ for $a=\sqrt{2}$; $\frac{5\pi}{4}+2\pi k$ for $a=-\sqrt{2}$; $\frac{\pi}{4}+\pi k$ for $\begin{cases} a\neq\sqrt{2} \\ a\neq-\sqrt{2}.\end{cases}$ **1678.** $\varnothing$ for $a=\frac{\pi}{4}+\frac{\pi}{2}n$; $x\neq\frac{\pi}{2}+\pi k$ for $a=\pi n$; $\frac{\pi}{4}+\frac{\pi}{2}k$ for $\begin{cases} a\neq\pi n \\ a\neq\frac{\pi}{4}+\frac{\pi}{2}m.\end{cases}$ **1679.** $\varnothing$ for $a=\frac{\pi}{2}+\pi n$, $\frac{3\pi}{4}+\pi n<a<\frac{5\pi}{4}+\pi n$; $\arctan\left(\tan a\pm\sqrt{\tan^2 a-1}\right)+\pi k$ for $\frac{\pi}{4}+\pi n\leqslant a<\frac{\pi}{2}+\pi n$, $\frac{\pi}{2}+\pi n<a\leqslant\frac{3\pi}{4}+\pi n$. **1680.** $\varnothing$ for $-2<a<2$; $\pm\arccos\frac{a+\sqrt{a^2-4}}{2}+2\pi k$ for $a\leqslant -2$; $\pm\arccos\frac{a-\sqrt{a^2-4}}{2}+2\pi k$ for $a\geqslant 2$. **1681.** $\varnothing$ for $\varphi+2\pi n<a<\pi-\varphi+2\pi n$; $\arctan\frac{\cos a\pm\sqrt{\cos^2 a-\sin a}}{\sin a}+\pi k$ for $-\pi-\varphi+2\pi n\leqslant a\leqslant\varphi+2\pi n$ $\left(\varphi=\arcsin\frac{-1+\sqrt{5}}{2}\right)$. **1682.** $\varnothing$ for $a=-1$; 1 for $a\neq -1$. **1683.** $\varnothing$ for $a\leqslant 0$, $a=1$; $\pm\sqrt{\frac{2}{a}}$ for $\begin{cases} a>0 \\ a\neq 1.\end{cases}$ **1684.** πk for

$a<-\frac{5}{4}$, $a>5$; $x_1=\pi k$, $x_2=\pm\arccos\frac{-1\pm\sqrt{4a+5}}{4}+2\pi k$ for $-\frac{5}{4}\leqslant a<1$; $x_1=\pi k$, $x_2=\pm\arccos\frac{-1+\sqrt{4a+5}}{4}+2\pi k$ for $1<a\leqslant 5$.

1685. $-\frac{\pi}{4}+\pi k$ for $a<-\frac{2}{3}$, $a>2$; $x_1=-\frac{\pi}{4}+\pi k$, $x_2=\frac{1}{2}(-1)^k\times \arcsin\frac{2a}{a+2}+\frac{\pi}{2}k$ for $-\frac{2}{3}\leqslant a\leqslant 2$. **1686.** $\varnothing$ for $a<\frac{-1-\sqrt{10}}{2}$, $a>\frac{-1+\sqrt{10}}{2}$; $x_1=\frac{\pi}{2}+\pi k$, $x_2=-\arctan 3+\pi k$ for $a=1$; $\arctan\frac{-1\pm\sqrt{-4a^2-4a+9}}{2(a-1)}+\pi k$ for $\frac{-1-\sqrt{10}}{2}\leqslant a<1$, $1<a\leqslant\frac{-1+\sqrt{10}}{2}$. **1687.** $\varnothing$ for $\begin{cases} b=2\pi k\\ a\neq 0;\end{cases}$ $(t;\ b-t)$, where $t\in\boldsymbol{R}$, for $\begin{cases} b=2\pi k\\ a=0;\end{cases}$ $\varnothing$ for $\begin{cases} b\neq 2\pi k\\ \left|\dfrac{a}{2\sin\frac{b}{2}}\right|>1;\end{cases}$ $\left(\frac{b+\alpha}{2}+2\pi k;\ \frac{b-\alpha}{2}-2\pi k\right)$, $\left(\frac{b-\alpha}{2}+2\pi k;\ \frac{b+\alpha}{2}-2\pi k\right)$, where $\alpha=2\arccos\dfrac{a}{2\sin\frac{b}{2}}$ for $\begin{cases} b\neq 2\pi k\\ \left|\dfrac{a}{2\sin\frac{b}{2}}\right|\leqslant 1.\end{cases}$ **1688.** $\varnothing$ for $\begin{cases} a\neq 0\\ b=2\pi k\end{cases}$ and for $\begin{cases} b\neq 2\pi k\\ \left|\dfrac{a}{2\sin\frac{b}{a}}\right|>1;\end{cases}$ $(t;\ b-t)$, where $t\in\boldsymbol{R}$, for $\begin{cases} a=0\\ b=2\pi k;\end{cases}$ $\left(\frac{b+\alpha}{2}+2\pi k;\ \frac{b-\alpha}{2}-2\pi k\right)$, $\left(\frac{b-\alpha}{2}+\pi+2\pi k;\ \frac{b+\alpha}{2}-\pi-2\pi k\right)$, where $\alpha=-2\arcsin\dfrac{a}{2\sin\frac{b}{2}}$ for $\begin{cases} b\neq 2\pi k\\ \left|\dfrac{a}{2\sin\frac{b}{2}}\right|\leqslant 1.\end{cases}$ **1689.** $\varnothing$ for $|2a+\cos b|>1$; $\left(\frac{b+\alpha}{2}+\pi k;\ \frac{b-\alpha}{2}-\pi k\right)$, $\left(\frac{b-\alpha}{2}+\pi k;\ \frac{b+\alpha}{2}-\pi k\right)$, where $\alpha=\arccos(2a+\cos b)$, for $|2a+\cos b|\leqslant 1$. **1690.** $\varnothing$ for $|2a-\sin b|>1$; $\left(\frac{b+\alpha}{2}+\pi k;\ \frac{b-\alpha}{2}-\pi k\right)$, $\left(\frac{\pi+b-\alpha}{2}+\pi k;\ \frac{-\pi+b+\alpha}{2}-\pi k\right)$, where $\alpha=\arcsin(2a-\sin b)$, for $|2a-\sin b|\leqslant 1$. **1691.** $\varnothing$ for $\begin{cases} a\neq 0\\ b=\pi k\end{cases}$ and for $\begin{cases} b\neq\pi k\\ \left|\dfrac{a}{\sin b}\right|>1\end{cases}$; $(t;\ b-t)$, where $t\in\boldsymbol{R}$, for $\begin{cases} b=\pi k\\ a=0;\end{cases}$ $\left(\frac{b+\alpha}{2}+\pi k;\right.$

$\frac{b-\alpha}{2}-\pi k\Big)$, $\left(\frac{\pi+b-\alpha}{2}+\pi k;\ \frac{-\pi+b+\alpha}{2}-\pi k\right)$, where $\alpha=\arcsin\frac{a}{\sin b}$, for $\begin{cases} b\neq\pi k \\ \left|\frac{a}{\sin b}\right|\leqslant 1 \end{cases}$. **1692.** $\varnothing$ for $a<-\frac{1}{4}$, $a>\frac{1}{4}$; $(\alpha+\pi(k+n);\ -\beta+\pi(k-n))$, $(\beta+\pi(k+n);\ -\alpha+\pi(k-n))$, $(-\beta+\pi(k+n);\ \alpha+\pi(k-n))$, $(-\alpha+\pi(k+n);\ \beta+\pi(k-n))$, where $\alpha=\frac{\arccos 4a+\arccos 2a}{2}$, $\beta=\frac{\arccos 4a-\arccos 2a}{2}$, for $-\frac{1}{4}\leqslant a\leqslant\frac{1}{4}$. **1693.** $\varnothing$ for $a<-\frac{1}{3}$, $a>\frac{1}{3}$; $(\alpha+\pi(k+n);\ \beta+\pi(k-n))$, $\left(\frac{\pi}{2}+\beta+\pi(k+n);\ -\frac{\pi}{2}+\alpha+\pi(k-n)\right)$, $\left(\frac{\pi}{2}-\beta+\pi(n+k);\ \frac{\pi}{2}-\alpha+\pi(n-k)\right)$, $(\pi-\alpha+\pi(n-k);\ \pi-\beta+\pi(n-k))$, where $\alpha=\frac{\arcsin 3a+\arcsin a}{2}$, $\beta=\frac{\arcsin 3a-\arcsin a}{2}$ for $-\frac{1}{3}\leqslant a\leqslant\frac{1}{3}$. **1694.** $\varnothing$ for $\begin{cases} a=0 \\ b\neq\pi k \end{cases}$ and for $\begin{cases} a\neq 0 \\ \left|\frac{2\sin b}{a}+\cos b\right|>1 \end{cases}$; $(t;\ b-t)$, where $t\in\boldsymbol{R}$, for $\begin{cases} a=0 \\ b=\pi k; \end{cases}$ $\left(\frac{b+\alpha}{2}+\pi k;\ \frac{b-\alpha}{2}-\pi k\right)$, $\left(\frac{b-\alpha}{2}+\pi k;\ \frac{b+\alpha}{2}-\pi k\right)$, where $\alpha=\arccos\left(\frac{2\sin b}{a}+\cos b\right)$, for $\begin{cases} a\neq 0 \\ \left|\frac{2\sin b}{a}+\cos b\right|\leqslant 1 \end{cases}$. **1695.** $\varnothing$ for $a\neq 0$; $\left(\frac{\pi}{2}+\pi(k+n);\ \frac{\pi}{2}(k-n)\right)$ for $a=0$. **1696.** $\varnothing$ for $a\neq\pm\frac{\pi}{3}+\pi n$; $\left(\frac{a}{2}+\pi k;\ -\frac{a}{2}+\pi k\right)$ for $a=\pm\frac{\pi}{3}+2\pi n$; $\left(\frac{\pi+a}{2}+\pi k;\ \frac{\pi-a}{2}+\pi k\right)$ for $a=\pm\frac{2\pi}{3}+2\pi n$. **1697.** $\varnothing$ for $a<-\frac{1}{2}$, $a>\frac{1}{2}$; $((-1)^{k+1}\arcsin a+\pi k;\ (-1)^n\arcsin 2a+\pi n)$, $((-1)^k\arcsin 2a+\pi k;\ (-1)^{n+1}\arcsin a+\pi n)$ for $-\frac{1}{2}\leqslant a\leqslant\frac{1}{2}$. **1698.** $\varnothing$ for $a<\frac{3}{2}$; $\operatorname{arccot}\left(-\frac{1}{2}\right)+\pi k$ for $a=\frac{3}{2}$; $\operatorname{arccot}\frac{-1+\sqrt{2a-3}}{2}+\pi k<x<\operatorname{arccot}\frac{-1-\sqrt{2a-3}}{2}+\pi k$ for $a>\frac{3}{2}$. **1699.** $\varnothing$ for $a<\frac{3}{2}$; $\frac{\pi-\alpha}{2}+\pi k$ for $a=\frac{3}{2}$; $\begin{cases} \frac{\beta-\alpha}{2}+\pi k\leqslant x\leqslant-\frac{\beta+\alpha}{2}+\pi k \\ x\neq\pi n \end{cases}$ for $a>\frac{3}{2}$ $\left(\alpha=\arccos\frac{a}{\sqrt{a^2+4}},\ \beta=\arccos\frac{a-4}{\sqrt{a^2+4}}\right)$. **1700.** $-\frac{\pi}{2}+2\pi k<x\leqslant-\arccos\frac{a+\sqrt{a^2+4}}{2}+2\pi k$, $\arccos\frac{a+\sqrt{a^2+4}}{2}+2\pi k\leqslant x<\frac{\pi}{2}+2\pi k$ for $a<0$; $\pi+2\pi n$, $-\frac{\pi}{2}+2\pi k<x<\frac{\pi}{2}+2\pi k$ for $a=0$; $-\frac{\pi}{2}+2\pi k<x<\frac{\pi}{2}+2\pi k$, $\arccos\frac{a-\sqrt{a^2+4}}{2}+2\pi k\leqslant x\leqslant 2\pi-\arccos\frac{a-\sqrt{a^2+4}}{2}+2\pi k$ for $a>0$.

In 1988 Mir Publishers will be issuing "*Solving Problems in Geometry*", which is intended for the students in the mathematical and physical mathematical faculties at teacher training institutes which was written by V.N. Litvinenko, A.G. Mordkovich, and VA. Gusev. The "Solving Problems in Geometry" is a sequel to this book. It will contain some 1500 problems on every topic of plane and solid geometry. Most of the problems are of intermediate difficulty and should be the basis of the teaching or exercise programme. The book was written so that it may either accompany a lecture course or used for private studies. At the beginning of each section, therefore, examples are given of problems and their solutions, and in several cases some general theoretical and methodological remarks are made.

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